GEODESICS AND SPANNING TREES FOR EUCLIDEAN FIRST-PASSAGE PERCOLATION

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Abstract. The metric $D_\alpha(q, q')$ on the set Q of particle locations of a homogeneous Poisson process on $\mathbb{R}^d$, defined as the infimum of $(\sum_i |q_i - q_{i+1}|^\alpha)^{1/\alpha}$ over sequences in Q starting with q and ending with q' (where $|\cdot|$ denotes Euclidean distance) has nontrivial geodesics when $\alpha > 1$. The cases $1 < \alpha < \infty$ are the Euclidean first-passage percolation (FPP) models introduced earlier by the authors while the geodesics in the case $\alpha = \infty$ are exactly the paths from the Euclidean minimal spanning trees/forests of Aldous and Steele. We compare and contrast results and conjectures for these two situations. New results for $1 < \alpha < \infty$ (and any d) include inequalities on the fluctuation exponents for the metric ($\chi \leq 1/2$) and for the geodesics ($\xi \leq 3/4$) in strong enough versions to yield conclusions not yet obtained for lattice FPP: almost surely, every semi-infinite geodesic has an asymptotic direction and every direction has a semi-infinite geodesic (from every q). For $d = 2$ and $2 \leq \alpha < \infty$, further results follow concerning spanning trees of semi-infinite geodesics and related random surfaces.

0. Introduction

There is an extensive literature (see [Y] for a survey) concerning combinatorial optimization in which some functional based on the Euclidean distances $|q - q'|$ between random points in $\mathbb{R}^d$ is minimized. Familiar examples include the total length in the travelling salesman problem and in the minimal spanning tree. In [HoN1], the authors introduced another family of such functionals in order to obtain Euclidean versions of the first-passage percolation (FPP) models originally defined in the context of the $\mathbb{Z}^d$ lattice by Hammersley and Welsh [HW]. (We remark that other Euclidean FPP models were introduced by Vahidi-Asl and Wierman [VW1, VW2] and studied by them and by Serafini [Se].) The

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focus of this paper is on these Euclidean FPP models from two perspectives. First, we survey a number of results and conjectures about these models with special emphasis on contrasts to the closely related but very different minimal spanning tree/forest of Aldous and Steele [AS]. Second, we derive a number of new results about Euclidean FPP, and explain why some of these go well beyond what has been proved for lattice FPP. It is our hope that the reader will find the pedagogical and research aspects of the paper to be complementary rather than antagonistic.

We define, for $r = (q_1, \ldots, q_k)$ a finite sequence of points in $\mathbb{R}^d$ and for $\alpha > 0$ (usually we take $\alpha > 1$),

$$T_\alpha(r) = \sum_{j=1}^{k-1} |q_j - q_{j+1}|^\alpha, \quad C_\alpha(r) = (T_\alpha(r))^{1/\alpha}; \tag{0.1}$$

for $\alpha = \infty$, we set

$$C_\infty(r) = \max\{|q_j - q_{j+1}| : 1 \le j < k\}. \tag{0.2}$$

Starting from some (random) set of points $\tilde{Q}$ in $\mathbb{R}^d$, we fix some q and q' in $\tilde{Q}$ and then consider the combinatorial optimization problem of obtaining $D_{\alpha,\tilde{Q}}(q,q') \equiv \inf\{C_\alpha(r)\}$ where the infimum is over all finite sequences r in $\tilde{Q}$ with $q_1 = q$ and $q_k = q'$ where k is the (arbitrary) length of r. When $\tilde{Q}$ is finite (e.g., N independent uniformly distributed points in a cube of volume N) there is of course some minimizing r that yields the infimum, but we will be interested in the case where $\tilde{Q}$ is a homogeneous Poisson point process on all of $\mathbb{R}^d$ (corresponding to $N \to \infty$) and then the issue of a minimizing r is less trivial.

This issue is closely related to that of the existence of a geodesic path between q and q' for the metric (when $\alpha \ge 1$) $D_{\alpha,\tilde{Q}}$. It turns out that the existence of such a geodesic between arbitrary points q and q' is no problem for the Euclidean FPP models where $1 \le \alpha < \infty$, but for Euclidean minimal spanning trees, which as we shall see correspond to $\alpha = \infty$, this is a serious issue which is not yet resolved for $d > 2$.

In the next section of the paper, we give precise definitions of geodesics (finite and

infinite), explain why finite geodesics between arbitrary q and q' always exist when $\alpha < \infty$ and why they may not exist when $\alpha = \infty$. We then review previous results for both lattice and Euclidean FPP and state our new results concerning the existence, nature and use of semi-infinite geodesics. The latter are based on new estimates concerning the two exponents, χ and ξ, describing respectively the fluctuation of the metric and of its geodesics. These estimates are presented (and their relation to related results for lattice FPP is discussed) in Section 2 and are used there to prove the new results of Section 1. In Sections 3 and 4, the fluctuation exponent estimates are proved. Some technical lemmas are given in Section 5.

1. Geodesics and Spanning Trees

Although our primary interest is in lattice and Euclidean FPP and Euclidean minimal spanning trees/forests, we will present the basic definitions in the general context of a countable set $\tilde{Q}$ (in our concrete examples, this will be a subset of $\mathbb{R}^d$ with $d \geq 2$) and a function $\tau : \tilde{Q} \times \tilde{Q} \to [0, \infty]$ (e.g., $\tau(q, q') = |q - q'|^\alpha$). We insist that $\tau(q, q) = 0$ for every $q \in \tilde{Q}$ and that $\tau(q, q') > 0$ when $q \neq q'$ (although this latter condition can be relaxed, e.g., in lattice FPP models). In our examples, $\tau(q, q') = \tau(q', q)$, but this would not be so in directed (or oriented) FPP models.

A path r is a sequence $(q_i : i \in I)$ that is indexed by an interval I in $\mathbb{Z}$; it is finite, semi-infinite or doubly-infinite according to the index set I. (For semi-infinite paths, we generally take I infinite to the right, i.e., of the form $(i_0 + 1, i_0 + 2, \dots)$.) We also define a segment of a path $r = (q_i : i \in I)$ to be any subpath $r' = (q_i : i \in J)$ with J a sub-interval of I. We call a path self-avoiding if $q_i \neq q_j$ for any $i \neq j \in I$.

To each $i \in I$ (such that $i + 1 \in I$) we associate $\tau_i = \tau(q_i, q_{i+1})$ and to each finite path $r = (q_{i_0+1}, \dots, q_{i_0+k})$ of length $k > 1$, we associate a cost function $\tilde{C}(r) =$

$\tilde{C}_k(\tau(q_{i_0+1}, q_{i_0+2}), \ldots, \tau(q_{i_0+k-1}, q_{i_0+k})) \in [0, \infty]$ that is subadditive: for $k' \geq k > 1$,

$$\tilde{C}(\tau_1, \ldots, \tau_{k-1}, \tau_k, \ldots, \tau_{k'}) \;\leq\; \tilde{C}(\tau_1, \ldots, \tau_{k-1}) + \tilde{C}(\tau_k, \ldots, \tau_{k'}). \tag{1.1}$$

(For a path r of length 0, we take $\tilde{C}(r) = 0$.) Equivalently, in terms of a path $r = (q, \ldots, \hat{q}, \ldots, q')$ from q to q' passing through $\hat{q}$ and thought of as the concatenation of $r_1 = (q, \ldots, \hat{q})$ and $r_2 = (\hat{q}, \ldots, q')$, we have

$$\tilde{C}(r) \;\leq\; \tilde{C}(r_1) + \tilde{C}(r_2). \tag{1.2}$$

We also assume that $\tilde{C}(\tau_1, \ldots, \tau_n) = \infty$ if and only if some $\tau_i = \infty$ and that $\tilde{C}(\tau_1, \ldots, \tau_n) = \tilde{C}(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_n)$ if $\tau_j = 0$. The examples we consider are: $\tilde{C} = \sum_i \tau_i^{\alpha}$ or $(\sum_i \tau_i^{\alpha})^{1/\alpha}$ (with $\tau(q, q') = |q - q'|$); $(\sum_i \tau_i)^{1/\alpha}$ (with $\tau(q, q') = |q - q'|^{\alpha}$); and $\max_i \tau_i$. Taking $\alpha \geq 1$ yields (1.1) and (1.2).

In the usual lattice FPP models (see, e.g., [Ke1]), $\tilde{Q} = \mathbb{Z}^d$ (or a subset of $\mathbb{Z}^d$) and $\tau(q, q') < \infty$ if and only if q and q' are nearest neighbors on $\mathbb{Z}^d$; for such pairs, the $\tau(q, q')$'s are i.i.d. random variables. In the Euclidean models of Vahidi-Asl and Wierman, $\tau(q, q') < \infty$ if and only if q and q' are neighboring points in the Voronoi or Delaunay graph associated with $\tilde{Q} \subset \mathbb{R}^d$. In our abstract setting one can define a graph G with vertex set $\tilde{Q}$ and edge set consisting of those $\{q, q'\}$ with $\tau(q, q') < \infty$. The assumption (1.1) (or equivalently (1.2)) is important because it yields the triangle inequality for a natural metric defined on each connected component of this graph as follows.

Definition. *Given $q, q' \in \tilde{Q}$, let $R(q, q')$ denote the set of all finite paths starting at q and ending at q' and define*

$$\tilde{D}(q, q') \;=\; \inf_{r \in R(q, q')} \tilde{C}(r) \tag{1.3}$$

(or ∞, if $R(q, q')$ is empty).

Note that $\tilde{D}(q, q) = 0$ and that (1.2) yields the triangle inequality,

$$\tilde{D}(q, q') \;\leq\; \tilde{D}(q, \hat{q}) + \tilde{D}(\hat{q}, q').$$

In our abstract setting, $\tilde{D}$ may not be a metric (but only a pseudometric) if in taking the infimum in (1.3), $\tilde{D}(q,q') = 0$ for some $q \neq q'$. This happens, for example, with $\tilde{C}(r) = (\sum_i |q_i - q_{i+1}|^2)^{1/2}$ if $\tilde{Q}$ is dense in $\mathbb{R}^d$. $\tilde{D}$ will in fact be a metric in all of our examples because $\tilde{C}_k(\tau_1, \ldots, \tau_{k-1}) \geq \tilde{C}_1(\tau_1)$ and

$$\inf_{q' \neq q} \tilde{C}_1(\tau(q,q')) > 0 \quad \text{for all } q \in \tilde{Q}. \tag{1.4}$$

Definition. *A finite path r starting at q and ending at q' is said to be minimizing if the infimum in (1.3) is finite and is achieved by r, i.e., if $\tilde{D}(q,q') = \tilde{C}(r) < \infty$. A (finite, semi-infinite or doubly infinite) path $r = (q_i : i \in I)$ is said to be a geodesic if it is self-avoiding and if every finite segment of r is minimizing.*

We note that in all our examples *except* those with $\tilde{C}(r) = \max_i \tau_i$, every self-avoiding minimizing finite path r is automatically a geodesic. This is because if $r_{(1)}$ were a non-minimizing segment of r, then representing r as a concatenation of $r_{(0)}$, $r_{(1)}$ and $r_{(2)}$, we could replace $r_{(1)}$ by an $r'_{(1)}$ (with the same endpoints as $r_{(1)}$ but with $\tilde{C}(r'_{(1)}) < \tilde{C}(r_{(1)})$) and thus obtain an r' with the same endpoints as r and with $\tilde{C}(r') < \tilde{C}(r)$ contradicting the minimizing property of r.

1.1 Euclidean FPP. In this subsection, we restrict attention to the Euclidean FPP models of [HoN1] where $\tilde{Q} = Q$, the set of particle locations of a homogeneous Poisson process of unit density on $\mathbb{R}^d$, and $\tilde{C}(r) = C_\alpha(r) = (\sum_j |q_j - q_{j+1}|^\alpha)^{1/\alpha}$ with $1 \leq \alpha < \infty$. We denote the corresponding metric $\tilde{D}$ by D_α. Within our general framework, one may (i) set $\tau(q,q') = |q - q'|$ and $\tilde{C}(\tau_1, \ldots, \tau_n) = (\sum_j \tau_j^\alpha)^{1/\alpha}$, or alternatively (ii) set $\tau(q,q') = |q - q'|^\alpha$ and $\tilde{C}(\tau_1, \ldots, \tau_n) = (\sum_j \tau_j)^{1/\alpha}$. (Indeed, as far as the geodesics are concerned, one could instead (iii) set $\tau(q,q') = |q - q'|^\alpha$ and $\tilde{C}(r) = T_\alpha(r) = \sum_j |q_j - q_{j+1}|^\alpha$ so that $\tilde{C}(\tau_1, \ldots, \tau_n) = \sum_j \tau_j$. The latter is best for comparing Euclidean FPP with lattice FPP while (i) is best for letting $\alpha \to \infty$ so that both $\tau(q,q')$ and $\tilde{C}(r)$ have a limit.)

When $\alpha = 1$ (or in version (iii) above when $0 < \alpha \leq 1$) and $d \geq 2$, since (almost surely) no three points of Q are collinear, it follows that for any distinct $q, q' \in Q$, the unique geodesic between them is the trivial one going from q to q' in one step — i.e., $r = (q, q')$. To get nontrivial geodesics we need $\alpha > 1$. The next proposition states that for $\alpha \neq \infty$, geodesics exist between all pairs of points (and are unique). It is the Euclidean analog of a standard result in lattice FPP (see, e.g., [SmW]) with a similar proof, which we sketch. Our focus for $\alpha < \infty$ will then be on the asymptotic behavior of the finite geodesic between q and q' as $|q - q'| \to \infty$ and on the existence, nature and abundance of infinite geodesics. As we shall see in the next subsection, when $\alpha = \infty$, even the existence of finite geodesics is nontrivial.

Proposition 1.1. *In Euclidean FPP with $d \geq 2$ and $1 \leq \alpha < \infty$, there is almost surely a unique geodesic $M_\alpha(q, q')$ between every pair of distinct points $q, q' \in Q$.*

Proof. The uniqueness follows because if r and r' were different self-avoiding paths from q to q' with $C_\alpha(r) = C_\alpha(r')$, there would be two disjoint sets $\{\{q_i, \bar{q}_i\} : i = 1, \ldots, m;\ q_i \neq \bar{q}_i\}$ and $\{\{q'_j, \bar{q}'_j\} : j = 1, \ldots, m';\ q'_j \neq \bar{q}'_j\}$ with $\sum_i |\bar{q}_i - q_i|^\alpha = \sum_j |\bar{q}'_j - q'_j|^\alpha$. But that occurs with zero probability.

To prove existence, note that the intersection of Q with the Euclidean ball $\mathcal{B}(0, K) \equiv \{x \in \mathbb{R}^d : |x| \leq K\}$ is, for any $K < \infty$, almost surely finite. We define for $q \in Q$,

$$d_\alpha(q, K) \;=\; \inf\{D_\alpha(q, q') : q' \in Q \setminus \mathcal{B}(0, K)\} \tag{1.5}$$

and claim that for every $q \in Q$,

$$\lim_{K \to \infty} D_\alpha(q, K) \;=\; \infty \;\; \text{a.s.} \tag{1.6}$$

This implies that for given q, q' and then some sufficiently large K, any r from q to q' that exits $\mathcal{B}(0, K)$ has $C_\alpha(r) > |q - q'|$ and hence the infimum in (1.3) must be achieved within the *finite* collection of (self-avoiding) paths staying in $\mathcal{B}(0, K)$. The claim (1.6) is

proved by appeal to a standard (continuum) percolation result (see [ZS1, ZS2]) — namely that for some sufficiently small $\epsilon > 0$, any semi-infinite self-avoiding path in Q must make infinitely many steps with $|q - q'| > \epsilon$. This easily yields (1.6). ■

As we shall see, analyzing the existence and nature of infinite geodesics can be difficult. But the following proposition, which shows that there is at least one semi-infinite geodesic starting from each $q \in Q$, is not hard.

Proposition 1.2. *Suppose $d \geq 2$ and $1 < \alpha < \infty$. For each $q \in Q$ define $R_\alpha(q)$ to be the graph with vertex set Q and edge set $\cup_{q' \in Q} M_\alpha(q, q')$. Almost surely, for every $q \in Q$, $R_\alpha(q)$ is a spanning tree on Q with every vertex having finite degree; thus there is at least one semi-infinite geodesic starting from every q.*

Proof. To see that $R_\alpha(q)$ is a spanning tree, order $Q = \{q_1, q_2, \ldots\}$ and note that inductively, for each n, $\cup_{i=1}^n M_\alpha(q, q_i)$ is a tree because of the uniqueness part of Proposition 1.1. To justify the finite degree claim (which we note is not valid when $\alpha = 1$), it suffices to show that for each $\tilde{q} \in Q$, there are a.s. only finitely many $\bar{q} \in Q$ such that the single step path $r = (\tilde{q}, \bar{q})$ is a geodesic. This is a consequence of

$$\lim_{K \to \infty} P[(\tilde{q}, \bar{q}) \text{ is a geodesic for some } \tilde{q}, \bar{q} \text{ with } |\tilde{q}| \leq 1,\, |\bar{q}| \geq K] \;=\; 0, \tag{1.7}$$

which itself follows from Lemma 5.2 (see (5.5)). We note that the key geometric idea here is to define

$$\mathcal{W}(a, b) \;=\; \{x \in \mathbb{R}^d : |a - x|^\alpha + |x - b|^\alpha \leq |a - b|^\alpha\} \tag{1.8}$$

and realize that $(\tilde{q}, \bar{q})$ cannot be a geodesic unless $\mathcal{W}^\circ(\tilde{q}, \bar{q})$, the interior of $\mathcal{W}(\tilde{q}, \bar{q})$, is devoid of Poisson particles. ■

When $\alpha = \infty$, there will also be at least one semi-infinite geodesic starting from every q. In that case, however, it is believed to be unique (see Conjecture 1 below)— unlike when $\alpha < \infty$ as we discuss later. A question apparently first posed (for lattice FPP) by

H. Furstenberg (see p. 258 of [Ke1]) is: What about doubly infinite geodesics? Here it is believed that a.s. these do not exist both for $\alpha < \infty$ and $\alpha = \infty$. We shall see later the extent to which this has been proved.

1.2 Minimal Spanning Trees and Forests. In this subsection (and the rest of the paper) we continue to take $\tilde{Q} = Q$, a homogeneous Poisson process of unit density (except as noted) on $\mathbb{R}^d$, but for now we take $\tilde{C}(r) = C_\infty(r) = \max_j |q_{j+1} - q_j|$ with the corresponding metric $\tilde{D}(q, q') = D_\infty(q, q')$, the minimax of $|q_{j+1} - q_j|$ along paths $r \in R(q, q')$.

Let us denote by $R^*(q, \bar{q})$ the set of all paths in $R(q, \bar{q})$ that do not use the edge $\{q, \bar{q}\}$. In order that the edge $\{q, \bar{q}\}$ belong to some geodesic, it is necessary and sufficient that $r = (q, \bar{q})$ is itself a geodesic and, a.s., this is true if and only if

$$|q - \bar{q}| < C_\infty(r) \text{ for every } r \in R^*(q, \bar{q}). \tag{1.9}$$

Following Alexander [Al2], we make the following

Definition. R_∞ *is the graph with vertex set* Q *and edge set consisting of those* $\{q, \bar{q}\}$*'s satisfying (1.9).*

The graph R_∞ can have no loops because on any loop, the edge $\{q, \bar{q}\}$ with maximum $|q - \bar{q}|$ does not satisfy (1.9). Thus R_∞ is a forest (a union of one or more disjoint trees) and contains at most one path between any q, q'. Every finite geodesic $M_\infty(q, q')$ must be a path in R_∞ and it is also not hard to see that every path in R_∞ is a geodesic. Thus, as in the $\alpha < \infty$ case, if a geodesic $M_\infty(q, q')$ exists between q and q' it will be unique; however, when $\alpha = \infty$, it may not exist. $M_\infty(q, q')$ exists if and only if q and q' are in the same connected component of R_∞ (if they are in different components, we set $M_\infty(q, q') = \emptyset$). Thus geodesics exist between every pair q, q' in Q if and only if R_∞ is a single (spanning) tree.

Definition. $R_\infty(q)$ *is the graph with vertex set* Q *and edge set* $\cup_{q' \in Q} M_\infty(q, q')$.

Clearly $R_\infty(q)$ is just the connected component of q in R_∞; it is not hard to see that each $R_\infty(q)$ must be an infinite tree. If R_∞ is a single tree, then (unlike when $\alpha < \infty$) the $R_\infty(q)$'s are all the *same* spanning tree.

It is shown in [Al2] that R_∞ is the same as the minimal spanning forest (MSF) constructed by Aldous and Steele [AS] as follows: For $K < \infty$, let R_∞^K denote the spanning tree of $Q \cap \mathcal{B}(0, K)$ that minimizes the sum of $|q - \bar{q}|$ over all edges $\{q, \bar{q}\}$ in the tree; then $R_\infty^K \to R_\infty$ as $K \to \infty$. There are two obvious qualitative issues concerning R_∞. Is it a single spanning tree or not? How many different semi-infinite geodesics start from q? This number, which is clearly the same for all q's in any fixed connected component of R_∞, equals the number of (topological) ends of the component. (An end is an equivalence class of semi-infinite paths in R_∞ that agree except for finite initial segments.) As to the number of ends, Alexander's results [Al2] combined with a natural conjecture about continuum percolation lead to the following (the natural conjecture is that at the critical radius R_c^* for overlapping balls of fixed radius centered at points of Q to form infinite clusters, there a.s. is no infinite cluster; for an extensive presentation of rigorous results about continuum percolation, see [MR]):

• **Conjecture 1 [Al2].** For any $d \geq 2$, R_∞ contains exactly one semi-infinite geodesic from each q.

Note that this includes the conjecture that there are no doubly infinite geodesics. The latter conjecture will persist for $\alpha < \infty$ even though Conjecture 1 will not. As to the other issue, the natural extension from the lattice case of a conjecture of Newman and Stein [NewS1, NewS2] is

• **Conjecture 2 [NewS1, NewS2].** For $d < 8$ (and perhaps also $d = 8$), R_∞ is a single spanning tree; for $d > 8$, R_∞ has (infinitely) many connected components.

The only dimension where these conjectures have been verified is $d = 2$, as stated in the next theorem. However we note that Conjecture 1 has been verified in lattice models

also for large d—see Example 2.7 of [Al2]. For general d, it has been proved [Al2] that at most one component of R_∞ has two ends and all others have a single end.

Theorem 1.3 [AlM, Al2]. *For $d = 2$, R_∞ is a single spanning tree with one end.*

In the next two subsections, we investigate the quite different qualitative nature of semi-infinite geodesics when $\alpha < \infty$. There will be many more infinite geodesics from each q and they will be asymptotically fairly regular. The irregularity of the infinite (or very long) paths in R_∞ is itself an interesting object of study. One way to pursue this issue is to consider for each x in $\mathbb{R}^d$ the (unique for $d = 2$ or under Conjecture 1) infinite path in R_∞ starting from (the q closest to) x, in the model with Poisson density $1/\delta^d$, as a random curve in $\mathbb{R}^d$ and study its subsequence limits in distribution as $\delta \to 0$. Some interesting results in this regard (especially for $d = 2$) have been obtained in [ABNW] using technical methods from [AB]. There are also interesting results on such scaling limits for other random spanning tree models in [ABNW] and [S].

1.3 Previous Results for Euclidean FPP. There are two types of previously known results. The first, valid for all d and $1 < \alpha < \infty$, concerns the asymptotic shape of large balls based on the metric D_α. The second, proved only for $d = 2$ and $2 \leq \alpha < \infty$, concerns semi-infinite geodesics $r = (q_1, q_2, \dots)$ with a specified asymptotic direction $\hat{x}$ — i.e., such that $q_k/|q_k| \to \hat{x} \in S^{d-1}$ as $k \to \infty$. We will call such an r an $\hat{x}$-geodesic. Doubly infinite geodesics $(\dots, q_{-1}, q_0, q_1, \dots)$ such that $q_k/|q_k| \to \hat{x}$ (resp., $\hat{y}$) as $k \to \infty$ (resp., $-\infty$) will be called $(\hat{x}, \hat{y})$-geodesics.

Both types of results were originally derived in [HoN1] as analogs of corresponding lattice FPP results. The first type differs from the lattice case in that the asymptotic shape is exactly a Euclidean ball (because of the statistical Euclidean invariance of the homogeneous Poisson point process). The significance of this difference for our new results will be discussed in Section 2 of this paper. We present the shape theorem result in a

slightly different form than the one of [HoN1]; in Section 2 (Theorem 2.3) we improve this result. For $x \in \mathbb{R}^d$, denote by $q(x)$ the Poisson particle location in Q closest to x (with any fixed rule for breaking ties). Then for $s > 0$, let $B_\alpha(x, s) \equiv \{q' \in Q : D_\alpha(q(x), q') \leq s\}$ denote the ball in Q of radius s centered at $q(x)$, using the metric D_α.

Theorem 1.4 [HoN1]. *For any $\alpha \in (1, \infty)$ and $d \geq 2$, there exists $\mu \in (0, \infty)$ depending on α and d, such that with $\mathcal{B}_0 \equiv \mathcal{B}(0, \mu^{-1})$ the following is true almost surely. For any $\epsilon \in (0, 1)$,*

$$Q \cap (1 - \epsilon)s\mathcal{B}_0 \ \subset \ B_\alpha(0, s^{1/\alpha}) \ \subset \ (1 + \epsilon)s\mathcal{B}_0 \tag{1.10}$$

for all sufficiently large s.

There are many natural questions one can ask about semi-infinite geodesics. We may focus on some $q \in Q$ (e.g., $q(0)$, the particle nearest the origin) and consider (for a fixed α), the set $G_\alpha(q)$ of semi-infinite geodesics starting from q. $G_\alpha(q)$ is of course just the set of semi-infinite paths starting from q in the spanning tree $R_\alpha(q)$ defined in Subsection 1.1 above, so that (for $1 < \alpha < \infty$, according to Proposition 1.2) $G_\alpha(q)$ is nonempty.

When $\alpha = \infty$, as discussed in Subsection 1.2, $R_\alpha(q)$ may not be spanning (for large enough d), but it is still an infinite tree of finite degree at each vertex, so $G_\infty(q)$ is also nonempty. For $\alpha = \infty$ and $d = 2$, according to Theorem 1.3, $G_\infty(q)$ consists of a single infinite geodesic and further for any q and q', the (unique) semi-infinite geodesics r and r' starting from q and q' coalesce; i.e., there is a unique $\bar{q} \in Q$ (which may be q or q') such that r (resp., r') is the concatenation of a path $\tilde{r}$ from q to $\bar{q}$ (resp., $\tilde{r}'$ from q' to $\bar{q}$) with the semi-infinite geodesic $\bar{r}$ starting from $\bar{q}$, while $\tilde{r}$ and $\tilde{r}'$ are disjoint except for $\bar{q}$. It is not hard to show (using the statistical rotational invariance of the Poisson point process) that here the semi-infinite geodesics cannot have an asymptotic direction (indeed, that the set of subsequence limit points of $q_k/|q_k|$ along a semi-infinite geodesic must a.s. be all of the unit circle).

But for $1 < \alpha < \infty$ and arbitrary d, one expects rather different answers to the following questions.

- **Question 1.** Does every semi-infinite geodesic have an asymptotic direction?

- **Question 2.** Is $R_\alpha(q, \hat{x})$, the set of $\hat{x}$-geodesics starting from q, nonempty for every unit vector $\hat{x}$?

- **Question 3.** Are there some $\hat{x}$'s with more than one $\hat{x}$-geodesic from some q (i.e., with $R_\alpha(q, \hat{x})$ bigger that a singleton)?

If the answer to Question 3 turns out to be "Yes", one may ask a related but different question, whose answer could still be "No", as follows.

- **Question 4.** For a deterministic $\hat{x}$, can there be more than one $\hat{x}$-geodesic from some q and can there be non-coalescing $\hat{x}$ geodesics from different q's?

- **Question 5.** Do doubly-infinite geodesics exist?

"Yes" answers to Questions 1, 2, and 3 are among the main new results of this paper and will be stated as theorems in the following subsection. Analogous results for lattice FPP are still open problems (see [New1] and Section 2 of this paper). The answer "No" to the fourth question was previously known, but only for restricted d and α (it remains an open problem in general), as follows; the restriction on α will be discussed below:

Theorem 1.5 [HoN1]. *For $d = 2$, $2 \leq \alpha < \infty$ and every deterministic $\hat{x}$, a.s. there is no more than one $\hat{x}$ geodesic from any q and a.s. any pair of $\hat{x}$ geodesics from distinct q, q' must coalesce.*

We remark that there are lattice FPP analogs to this theorem (and the next), but these have not been proved for *every* $\hat{x}$ [LN]; the best such result is due to Zerner (Theorem 1.5 in [New2]). As a consequence of the last theorem, there was a partial answer to Question 5, stated as the next theorem. The natural conjecture is that the correct answer to Question 5 is "No", certainly for $d = 2$ and perhaps for all d. (See Chap. 1 of [New2] for a discussion

of this conjecture for lattice FPP and its equivalence (when $d = 2$) to nonexistence of nonconstant ground states for disordered Ising ferromagnets. Other results in the lattice context are in [W].)

Theorem 1.6 [HoN1]. *For $d = 2$, $2 \leq \alpha < \infty$, and every deterministic $\hat{x}$ and $\hat{y}$, a.s. there are no $(\hat{x}, \hat{y})$-geodesics.*

An improvement of Theorem 1.6 (see Theorem 1.11) will be given below, basically as a consequence of our answer to Question 1, but this improvement falls well short of the conjecture that doubly infinite geodesics a.s. do not exist.

Behind the restriction to $\alpha \geq 2$ in Theorem 1.5 and 1.6 is the following lemma (Lemma 5 of [HoN1]), which we will use later.

Lemma 1.7 [HoN1]. *Suppose $r = (\ldots, q_i, q_{i+1}, \ldots)$ and $r' = (\ldots, q'_j, q'_{j+1}, \ldots)$ are two finite or infinite geodesics such that the closed line segments $\overline{q_i q_{i+1}}$ and $\overline{q'_j q'_{j+1}}$ intersect. If $d = 2$ and $2 \leq \alpha < \infty$, then $\{q_i, q_{i+1}\}$ and $\{q'_j, q'_{j+1}\}$ have at least one point in common.*

1.4 New Results on Infinite Geodesics for Euclidean FPP. The next three theorems, among the main new results of this paper, are consequences of fluctuation theorems presented in Section 2. The fluctuation theorems are of interest in their own right.

Theorem 1.8. *For $d \geq 2$, and $1 < \alpha < \infty$, a.s.: every semi-infinite geodesic has an asymptotic direction.*

Theorem 1.9. *For $d \geq 2$ and $1 < \alpha < \infty$, a.s.: for every $q \in Q$ and every unit vector $\hat{x}$, there is at least one $\hat{x}$-geodesic starting from q.*

Theorem 1.10. *For $d \geq 2$ and $1 < \alpha < \infty$, a.s.: for every $q \in Q$, the set $V(q)$ of unit vectors $\hat{x}$ such that there is more than one $\hat{x}$-geodesic starting at q is dense in the unit sphere.*

Remark. for $d = 2$, is is not hard to show (by arguments like those used to prove

Theorem 0 of [LN]) that a.s. $V(q)$ is countable. In general, whenever the answer to the first part of Question 4 is "No", then by an application of Fubini's Theorem, the Lebesgue measure (on the unit sphere S^{d-1}) of $V(q)$ is zero. But the proof of Theorem 1.10 also shows that $V(q)$ must have Hausdorff dimension at least $d-2$.

Theorem 1.8 implies that every doubly-infinite geodesic must be an $(\hat{x},\hat{y})$-geodesic for some $\hat{x},\hat{y} \in S^{d-1}$. But the proof of Theorem 1.8 implies a bit more. We state this in the next theorem in combination with the result of Theorem 1.6.

Theorem 1.11. *For $d \geq 2$ and $1 < \alpha < \infty$, a.s. doubly infinite geodesics other than $(\hat{x},-\hat{x})$-geodesics do not exist. In addition, for $d = 2$ and $2 \leq \alpha < \infty$, and any deterministically chosen $\hat{x}$, a.s. $(\hat{x},-\hat{x})$-geodesics do not exist.*

Theorem 1.11 is a step in the direction of verifying the conjecture that, a.s., doubly infinite geodesics do not exist. However, even for $d = 2$ and $2 \leq \alpha < \infty$, it does not prove the conjecture since it leaves open the possible existence of $(\hat{x},-\hat{x})$-geodesics with $\hat{x}$ dependent on the realization of Q.

In the rest of this subsection, we restrict attention to $2 \leq \alpha < \infty$ and $d = 2$ and explore some consequences of combining Theorems 1.5-1.11. This is in the spirit of [New1] (see Theorem 1.1 of that reference and the preceding discussion there), where the same issues were addressed, but only partially resolved, in the lattice FPP context.

When $d = 2$, $2 \leq \alpha < \infty$, $q \in Q$, and $\hat{x}$ is a deterministic unit vector (in S^1), by Theorems 1.5 and 1.9, there a.s. exists a unique $\hat{x}$-geodesic starting from q. We denote this semi-infinite geodesic by $s_q(\hat{x})$. In analogy with $R_\alpha(q)$, as defined in Proposition 1.2, (but with q replaced by "a point at infinity reached in the direction $\hat{x}$") we define $R_\alpha(\hat{x})$ to be the graph with vertex set Q and every edge contained in $\cup_{q' \in Q} s_{q'}(\hat{x})$. It follows from Theorem 1.5 that (a.s.) $R_\alpha(\hat{x})$ is a spanning tree on Q (the coalescing part of Theorem 1.5 ensures that $R_\alpha(\hat{x})$ has a single connected component). Since every edge in $R_\alpha(\hat{x})$

touching q is part of some geodesic, these edges belong to $R_\alpha(q)$ and hence, by Proposition 1.2, each vertex in $R_\alpha(\hat{x})$ has finite degree. We combine these facts with a few others in the following.

Theorem 1.12. *Suppose $d = 2$, $2 \leq \alpha < \infty$, and $\hat{x}$ is a deterministic unit vector (in S^1). Then the following are all valid a.s.. For any $q \in Q$ and any $\bar{q}_1, \bar{q}_2, \cdots \in Q$ such that $\bar{q}_k/|\bar{q}_k| \to \hat{x}$, the finite geodesic $M_\alpha(q, \bar{q}_k)$ converges as $k \to \infty$ to the unique $\hat{x}$-geodesic $s_q(\hat{x})$ starting from q. Thus the spanning trees $R_\alpha(\bar{q}_k) \to R_\alpha(\hat{x})$ as $k \to \infty$ (where the edges of $R_\alpha(\hat{x})$, as defined above, are those in $\cup_{q \in Q} s_q(\hat{x})$). $R_\alpha(\hat{x})$ is a spanning tree on Q (with every vertex having finite degree and) with a single infinite path from each q (namely $s_q(\hat{x})$); $R_\alpha(\hat{x})$ thus has a single topological end.*

Proof of Theorem 1.12. The things that remain to be proved are that $M_\alpha(q, \bar{q}_k) \to s_q(\hat{x})$ and that $R_\alpha(\hat{x})$ contains no infinite path from q other than $s_q(\hat{x})$. For a small $\epsilon > 0$, let $\hat{x}_+(\epsilon)$ (resp., $\hat{x}_-(\epsilon)$) be the unit vector obtained by rotating $\hat{x}$ by an angle ϵ in the clockwise (resp., counterclockwise) direction. By Theorems 1.5 and 1.9, there a.s. exist unique semi-infinite geodesics $s_q(\hat{x}_\pm(\epsilon))$ starting from q. For a path $r = (q_1, q_2, \dots)$ let us denote by $\bar{r}$ the union of the line segments $\overline{q_i\, q_{i+1}}$ (as a subset of $\mathbb{R}^2$). The paths $s_q(\hat{x}_+(\epsilon))$ and $s_q(\hat{x}_-(\epsilon))$ bifurcate at some q' (perhaps equal to q) and then, by uniqueness of finite geodesics, have no further Q-particles in common. By Lemma 1.6, the sets $\overline{s_q(\hat{x}_+(\epsilon))}$ and $\overline{s_q(\hat{x}_-(\epsilon))}$ bifurcate at q' and have no further $\mathbb{R}^2$-points in common. Thus $\mathbb{R}^2 \backslash \{(\overline{s_q(\hat{x}_+(\epsilon))} \cup \overline{s_q(\hat{x}_-(\epsilon))}\}$ consists of two connected components (one "inside" and one "outside") that we will denote by $S_q^{\text{in}}(\hat{x}, \epsilon)$ and $S_q^{\text{out}}(\hat{x}, \epsilon)$. The inside (resp., outside) component is characterized by containing sequences $x_1, x_2, \dots$ in $\mathbb{R}^2$ such that $|x_j| \to \infty$ while the angle between $x_j/|x_j|$ and $\hat{x}$ converges to a point in $(-\epsilon, \epsilon)$ (resp., to a point outside $[-\epsilon, \epsilon]$).

Now, by Lemma 1.6 again (and the uniqueness of finite geodesics) once k is large enough that $\bar{q}_k \in S_q^{\text{in}}(\hat{x}, \epsilon)$, $\overline{M_\alpha(q, \bar{q}_k)}$ (except for its initial portion from q to q') must be entirely

within the closure of $S_q^{\text{in}}(\hat{x}, \epsilon)$ and thus the same must be true for any (subsequence) limit $\tilde{r}$ of $M_\alpha(q, \bar{q}_k)$. Since this is true for every $\epsilon > 0$, it follows that such an $\tilde{r}$ (which is automatically a geodesic starting from q) must be an $\hat{x}$-geodesic. But then by Theorem 1.5, $\tilde{r}$ is a.s. $s_q(\hat{x})$ as claimed.

Next suppose that $\hat{r}$ is an infinite path in $R_\alpha(\hat{x})$ starting from q and different than $s_q(\hat{x})$. We show that this leads to a contradiction. The path $\hat{r}$ must bifurcate from $s_q(\hat{x})$ at some q' (possibly with $q' = q$) with no further Q-particles in common. For any q'' on $\hat{r}$ after q', the concatenation of the segment of $\hat{r}$ from q'' to q' and the infinite segment of $s_q(\hat{x})$ starting at q' (which is just $s_{q'}(\hat{x})$) must be $s_{q''}(\hat{x})$ since $s_{q''}(\hat{x})$ and $s_q(\hat{x})$ must coalesce somewhere and if it were not at q', $R_\alpha(\hat{x})$ would contain a loop. Let q''_k denote an infinite sequence of distinct such q'''s from $\hat{r}$ and let r'' be a limit of $s_{q''_k}(\hat{x})$ — which must exist since each $s_{q''_k}(\hat{x})$ passes through q' and contains $s_{q'}(\hat{x})$. Then r'' is a doubly infinite geodesic containing $s_{q'}(\hat{x})$ and thus by the first part of Theorem 1.11 is an $(\hat{x}, -\hat{x})$-geodesic. But this contradicts the second half of Theorem 1.11, which completes the proof. ■

Now that we have constructed in Theorem 1.12 the spanning tree $R_\alpha(\hat{x})$ composed of the $\hat{x}$-geodesics $s_q(\hat{x})$, we may ask: what is it good for? Following [New1], it can be used to study the surface of large balls in the metric space (Q, D_α) by means of certain (random) "height functions" on Q (or on $\mathbb{R}^d$). For a fixed $\alpha < \infty$ we replace D_α by the pseudometric on $\mathbb{R}^d$, $T_\alpha(x, y) \equiv D_\alpha(q(x), q(y))^\alpha$ (where $q(x)$ is the closest $q \in Q$ to x) and look at the pseudometric balls, $\tilde{B}_\alpha(x, s) \equiv \{y \in \mathbb{R}^d : T_\alpha(x, y) \leq s\}$. These are unions of Voronoi regions and are related to the balls $B_\alpha(x, s)$ for the metric D_α (defined just above Theorem 1.4) by $B_\alpha(x, s^{1/\alpha}) = \tilde{B}_\alpha(x, s) \cap Q$.

What does a large-radius ball $\tilde{B}_\alpha(x, s)$ look like when "viewed from its surface?" A natural interpretation of this question, that places the surface near the origin, is to consider the limit of $\tilde{B}_\alpha(\bar{q}_k, T_\alpha(\bar{q}_k, 0))$ as $|\bar{q}_k| \to \infty$ with $\bar{q}_k / |\bar{q}_k| \to \hat{x}$. Theorem 1.12 allows us to analyze this limit in terms of a function $H^{\hat{x}}(q, q')$ on $Q \times Q$ defined as follows. For $q, q' \in Q$,

define $W_{\hat{x}}(q,q')$ as the unique q'' in Q where $s_{\hat{x}}(q)$ and $s_{\hat{x}}(q')$ coalesce ($W_{\hat{x}}$ might be q or q') so that the path in $R_\alpha(\hat{x})$ between q and q' is the concatenation of $M_\alpha(q, W_{\hat{x}}(q,q'))$ and $M_\alpha(W_{\hat{x}}(q,q'),q')$. The following is mostly a consequence of Theorem 1.12.

Theorem 1.13. *Suppose $d=2$, $2 \le \alpha < \infty$, and $\hat{x}$ is a deterministic direction. Then the following are valid a.s.: For all $q,q' \in Q$ and any $\bar{q}_1, \bar{q}_2, \cdots \in Q$ such that $\bar{q}_k/|\bar{q}_k| \to \hat{x}$,*

$$H^{\hat{x}}(q,q') \;\equiv\; \lim_{k\to\infty}[T_\alpha(q,\bar{q}_k) - T_\alpha(q',\bar{q}_k)]$$

exists and equals $T_\alpha(q, W_{\hat{x}}(q,q')) - T_\alpha(q', W_{\hat{x}}(q,q'))$. The balls $\tilde{B}_\alpha(\bar{q}_k, T_\alpha(\bar{q}_k, 0))$ converge as $k \to \infty$ to $\{y \in \mathbb{R}^d : H^{\hat{x}}(q(y), q(0)) \le 0\}$. Furthermore, $H^{\hat{x}}(\cdot, q(0))$ as a function on Q satisfies

$$H^{\hat{x}}(q,q(0)) \;=\; \inf_{q'\neq q}[|q-q'|^\alpha + H^{\hat{x}}(q',q(0))] \tag{1.11}$$

and more generally for Q_0 any finite subset of Q containing q

$$H^{\hat{x}}(q,q(0)) \;=\; \inf_{q'\in Q\setminus Q_0}[T_\alpha(q,q') + H^{\hat{x}}(q',q(0))] \tag{1.12}$$

Proof. The only claims that require any explanation are (1.11) and (1.12). To prove (1.11), we let q'' denote the first particle after q on $s_{\hat{x}}(q)$ and note that by Theorem 1.12,

$$\begin{aligned} H^{\hat{x}}(q,q(0)) \;&=\; \lim_{k\to\infty}[|q-q''|^\alpha + T_\alpha(q'',\bar{q}_k) - T_\alpha(q(0),\bar{q}_k)] \\ &=\; |q-q''|^\alpha + H^{\hat{x}}(q'',q(0)). \end{aligned} \tag{1.13}$$

This bounds $H^{\hat{x}}(q,q(0))$ below by the right side of (1.11). The opposite inequality easily follows from $T_\alpha(q,\bar{q}_k) \ge \inf_{q'\neq q}(|q-q'|^\alpha + T_\alpha(q',\bar{q}_k))$. The identity (1.12) is derived by quite similar arguments to those used for (1.11). ■

We now consider the random field $H^{\hat{x}}(q(y),q(0))$. It is clear, at least on a heuristic level, that the asymptotic behavior of its mean, as $|y| \to \infty$, is $-\mu(\alpha,2)(\hat{x}\cdot y)$ to leading order, where $\mu(\alpha,d)$ is the inverse of the radius appearing in Theorem 1.4 and $\hat{x}\cdot y$ denotes the

standard Euclidean inner product. When $\hat{x} \cdot y \neq 0$, it seems reasonable that the variance of $H^{\hat{x}}(q(y), q(0))$ should have a leading order behavior similar to that of $T_\alpha(0, y)$ — namely like $|y|^{2\chi}$ (with $\chi = 1/3$ conjectured for $d = 2$, as discussed in Section 2). For $\hat{x} \cdot y = 0$, where by symmetry $E[H^{\hat{x}}(q(y), q(0)] = 0$, it seems that for $d = 2$, one should expect the variance to grow faster, namely linearly in $|y|$, and correspondingly the boundary of the region where $H^{\hat{x}}(q(x), q(0)) \leq 0$ should fluctuate from the straight line $y = t\hat{y}_0$ (where $\hat{y}_0 \cdot \hat{x} = 0$) by a distance of order $\sqrt{t}$ (see, e.g., [KrS]). This is related to the conjectured identity $\xi = 2\chi$ (for $d = 2$) for the fluctuation exponents ξ and χ that are the main topic of the next section. We remark that the exact values $\chi = 1/3$ and $\xi = 2/3$ have been derived recently in [BDJ, J] for a model related to random permutations, one of whose many guises is a kind of $d = 2$ directed FPP.

There are many interesting open questions one can ask about height functions on Q satisfying (1.11) and (1.12), such as whether there exist ones essentially different from those of the form $H^{\hat{x}}(q, q(0))$. For example, in general d one could take two deterministic sequences of points $\bar{q}_k^{(1)}$ and $\bar{q}_k^{(2)}$ with $|\bar{q}_k^{(1)}| = |\bar{q}_k^{(2)}| \to \infty$ and with $\bar{q}_k^{(j)}/|\bar{q}_k^{(j)}| \to \hat{x}^{(j)}$ for $j = 1, 2$ as $k \to \infty$ and then study

$$\min_{j=1,2}(T_\alpha(q, \bar{q}_k^{(j)})) - \min_{j=1,2}(T_\alpha(q', \bar{q}_k^{(j)})) \tag{1.15}$$

as $k \to \infty$. It could be (and this seems likely the case for $d = 2$) that the limit (in distribution) of this random function of q and q' is a symmetric mixture of the distributions of $H^{\hat{x}^{(1)}}$ and $H^{\hat{x}^{(2)}}$. This would be because the boundary between the region of Q where $T_\alpha(q, \bar{q}_k^{(1)}) < T_\alpha(q, \bar{q}_k^{(2)})$ and where $T_\alpha(q, \bar{q}_k^{(1)}) > T_\alpha(q, \bar{q}_k^{(2)})$ would (probably) be far from the origin as $k \to \infty$. On the other hand, it is conceivable (e.g., for large enough d, if $\chi = 0$ there—see the discussion and references in [KrS] or [NewP]) that this boundary would not wander off to infinity but rather would have an a.s. limit, and thus that (1.15) would also have a limit. The latter limit, defined for all q, q', should equal either $H^{\hat{x}^{(1)}}$ or $H^{\hat{x}^{(2)}}$, but

only when q and q' are both on the same side of the limit boundary. Thanks to Theorem 1.12, we can now pose such questions, but answering them remains a task for the future.

2. Fluctuation Results

Throughout this section and the remainder of the paper we deal with some fixed $d \geq 2$ and $\alpha \in (1, \infty)$. Occasionally, as noted, we will restrict our attention to $d = 2$ and $\alpha \in [2, \infty)$. We drop the α subscript in the (pseudo-) metric $T_\alpha(x, y) = T_\alpha(q(x), q(y))$ and the geodesic $M_\alpha(x, y) = M_\alpha(q(x), q(y))$ and denote these by $T(x, y)$ and $M(x, y)$. This section is organized as follows. In Subsection 2.1, we state two theorems giving large deviation bounds on $T(x, y)$ as $|x - y| \to \infty$; the proofs are given later in Sections 3 and 4. The first theorem concerns fluctuations about the mean and the second concerns fluctuations about $\mu|x - y|$. Here $\mu = \lim_{|x-y|\to\infty} ET(x, y)/|x - y|$ and also equals the a.s. limit of $T(0, n\hat{e})/n$ as $n \to \infty$ for any fixed unit vector $\hat{e}$ [HoN1]; it is of course the same μ appearing in the Shape Theorem 1.4. A third theorem in Subsection 2.1 gives a strengthened shape theorem like the one obtained for lattice FPP in [Al3, Ke2]. In Subsection 2.2, we state and prove (using the theorem about $T(x, y) - \mu|x - y|$) results about fluctuations of $M(x, y)$ from a straight line as $|x - y| \to \infty$. These results tell us that, with high probability, long finite geodesics (a) do not deviate far from the straight line between their endpoints and (b) do not start off in one direction and then "noticeably" change course. In Subsection 2.3, we apply these fluctuation results to prove Theorems 1.8-1.11.

2.1 Fluctuation of the Metric. In this subsection we consider fluctuations of $T_\ell \equiv T(0, \ell\hat{e}_1)$ where $\ell > 0$ and $\hat{e}_1$ is the unit vector $(1, 0, \ldots, 0)$. As in the case of lattice models, one expects that the standard deviation of T_ℓ grows like ℓ^χ for some exponent $\chi = \chi(d)$ that should not depend on α. For lattice FPP on $\mathbb{Z}^1$ (with ℓ an integer), the analog of T_ℓ is the sum of ℓ i.i.d. random variables ($\tau(j - 1, j)$ with $1 \leq j \leq \ell$) so that

$\chi(1) = 1/2$ (assuming $E[\tau(j-1,j)^2] < \infty$). For Euclidean FPP on $\mathbb{R}^1$, again $\chi(1) = 1/2$ although the argument, while standard, is not as trivial since T_ℓ then is essentially $\sum_{i=1}^{N} U_i^{\alpha}$ where the U_i are i.i.d. exponential random variables and N is random such that $\sum_{i=1}^{N} U_i$ is close to ℓ. For $d = 2$, $\chi(2)$ is believed to equal $1/3$ (see [HuH, K, HuHF, KPZ]), but the only models for which this (and much more) has been proved are certain directed FPP-like models related to random permutations (see [BDJ, J]). For lattice FPP with $d \geq 2$, there have been rigorous bounds on $\chi(d)$ including Kesten's result that $\chi(d) \leq 1/2$ [Ke2]. This latter bound has been strengthened by Kesten [Ke2] and Alexander [Al1, Al3] to give large deviation upper bounds for the deviation of T_ℓ as $\ell \to \infty$ from its mean and from the asymptotic expression $g(\ell\hat{e}_1)$ (or more generally for the deviation of $T(q,q')$ for $q, q' \in \mathbb{Z}^d$ as $|q - q'| \to \infty$ from its mean and from $g(q - q')$), where

$$g(v) \;=\; \lim_{n\to\infty} \frac{E[T(0,nv)]}{n}. \tag{2.1}$$

In the case of lattice FPP, g is a norm on $\mathbb{R}^d$ whose unit ball arises in the shape theorem [R, CD, Ke1, Bo]. For Euclidean FPP,

$$\lim_{\ell\to\infty} \frac{E[T(0,\ell\hat{x})]}{\ell} \;=\; \lim_{\ell\to\infty} \frac{ET_\ell}{\ell} \;=\; \mu \;\in\; (0,\infty) \tag{2.2}$$

(see (7) of [HoN1]), where $\mu = \mu(\alpha, d)$ appears in the shape theorem (Theorem 1.4) above. The next two theorems are the analogs for Euclidean FPP of the Kesten and Alexander results of [Al3, Ke2] for lattice FPP. The great advantage of Euclidean FPP over the lattice case is that the unit ball of the metric $g(v)$ (about which very little has been proved) is replaced by the *Euclidean* ball (of radius μ^{-1}). This allows us in the next subsection to go well beyond what was proved for lattice FPP, as we discuss there.

Here and for the remainder of the paper, we use C_0 to represent a strictly positive constant, to be thought of as small, that depends on α and d but never on ℓ. The actual value of C_0 may decrease as the paper progresses (perhaps even in a single line); all

statements made involving C_0 are valid with any smaller choice of C_0. Analogously, C_1 is a positive finite constant, thought of as large, whose value does not depend on ℓ but increases (with similar impunity) as the paper progresses. Certain other constants, appearing as exponents, we keep track of more carefully. We record their values here for easy reference:

$$\begin{aligned} \kappa_1 &= \min(1, d/\alpha), \\ \kappa_2 &= 1/(4\alpha+3), \\ \kappa_3 &= 1/(2\alpha), \\ \kappa_4 &= d/\alpha, \text{ and} \\ \kappa_5 &= 1/(4\alpha+2). \end{aligned} \tag{2.3}$$

Theorem 2.1. *Let $d \geq 2$ and $\alpha > 1$. For some constant C_1,*

$$Var T_\ell \;\leq\; C_1 \ell \;\text{ for }\; \ell \geq 0. \tag{2.4}$$

Additionally, with $\kappa_1 = \min(1, d/\alpha)$, $\kappa_2 = 1/(4\alpha+3)$, and for some constants C_0 and C_1,

$$P[\,|T_\ell - ET_\ell| > x\sqrt{\ell}] \;\leq\; C_1 \exp(-C_0 x^{\kappa_1}) \;\text{ for }\; \ell \geq 0 \;\text{ and }\; 0 \leq x \leq C_0 \ell^{\kappa_2}. \tag{2.5}$$

The proof of Theorem 2.1 is given in Section 3. The next theorem, which is essentially a replacement of ET_ℓ by $\mu\ell$ in (2.5), is proved in Section 4, by using Theorem 2.1 to show that

$$|ET_\ell - \mu\ell| \;\leq\; C_1 \sqrt{\ell} (\log \ell)^{1/\kappa_1}. \tag{2.6}$$

Theorem 2.2. *Let $d \geq 2$, $\alpha > 1$, $\kappa_1 = \min(1, d/\alpha)$ and $\kappa_2 = 1/(4\alpha+3)$. For any ϵ in $(0, \kappa_2)$, there exist constants C_0 and C_1 (depending on ϵ) such that*

$$P[|T_\ell - \mu\ell| \geq \lambda] \;\leq\; C_1 \exp(-C_0 (\lambda/\sqrt{\ell})^{\kappa_1}) \;\text{ for }\; \ell > 0 \;\text{ and }\; \ell^{\frac{1}{2}+\epsilon} \leq \lambda \leq \ell^{\frac{1}{2}+\kappa_2-\epsilon}. \tag{2.7}$$

A corollary of Theorem 2.2, the proof of which we sketch in Section 4, is the following improvement of Theorem 1.4; it is an analog of the Alexander-Kesten improved shape theorem for lattice FPP [Al3]:

Theorem 2.3. *For any $\alpha \in (1,\infty)$ and $d \geq 2$, with $\mathcal{B}_0 \equiv \mathcal{B}(0,\mu^{-1})$, the following is true almost surely:*

$$Q \cap \Big(1 - \frac{(\log s)^{2/\kappa_1}}{\sqrt{s}}\Big) s\mathcal{B}_0 \ \subset \ B_\alpha(0, s^{1/\alpha}) \ \subset \ \Big(1 + \frac{(\log s)^{2/\kappa_1}}{\sqrt{s}}\Big) s\mathcal{B}_0$$

for all sufficiently large s.

We make no claims about the optimality of the exponents κ_1 and κ_2 appearing in (2.5)-(2.7). We also note that the power $2/\kappa_1$ in Theorem 2.3 can be replaced by $(1+\epsilon)/\kappa_1$ with any $\epsilon > 0$. For lattice FPP with an exponential tail assumption on the underlying $\tau(q,q')$ variables, the analogous results in [Al3,Ke2] have $\kappa_1 = 1 = \kappa_2$. In the next subsection, we use Theorem 2.2 to control deviations of long finite geodesics from approximately straight line behavior.

2.2 Fluctuations of Geodesics. We want to use Theorem 2.2 to bound the probability that the geodesic $M(x,y)$ touches a Poisson particle located far from the straight line segment $\overline{xy}$. Our reasoning will follow that used in [New1] (see (3.2) there) but modified for the Euclidean context. We use (2.7) and some other arguments to show that for any $\epsilon > 0$, with high probability for large $|x-y|$, $M(x,y)$ does not deviate more than order $|x-y|^{\frac{3}{4}+\epsilon}$ from $\overline{xy}$. The wandering exponent $\xi = \xi(d)$ may be regarded as defined so that $|y-x|^\xi$ is the actual order of the typical (or largest) deviation from $\overline{xy}$. Thus, our next theorem implies that $\xi \leq 3/4$. It is conjectured that $\xi(2) = 2/3$ and decreases to $1/2$ for increasing d (see the discussion and references in [KrS] or [NewP]). A related result was obtained in [NewP] for lattice FPP but it was much weaker because of lack of information about the asymptotic shape $\mathcal{B}_0$ for lattice FPP. Roughly speaking, the lattice result was only valid when $y-x$ points in a direction where the boundary of $\mathcal{B}_0$ is curved. If it were proved that in a lattice model $\mathcal{B}_0$ is uniformly curved, then a result like the next theorem (which is only for *Euclidean* FPP) would follow — see [NewP] for details.

We define $M_\ell = M(0, \ell \hat{e}_1)$ and, for $A \subset \mathbb{R}^d$,

$$d_{max}\,(M_\ell, A) \;=\; \sup_{q \in M_\ell} \text{Dist}\,(q, A), \tag{2.8}$$

where $\text{Dist}\,(q, A)$ denotes the ordinary Euclidean distance from q to the set A. This represents the maximal Euclidean distance of (any point in) M_ℓ from A; if M_ℓ is replaced by a single point y, then $d_{\max}(y, A)$ is the usual Euclidean distance of y to the set A.

Theorem 2.4. *Let $d \geq 2$, $\alpha > 1$, $\kappa_1 = \min(1, d/\alpha)$ and $\kappa_2 = 1/(4\alpha + 3)$. For any $\epsilon \in (0, \kappa_2/2)$, there exist constants C_0 and C_1 (depending on ϵ) such that*

$$P[d_{\max}(M_\ell, \overline{0\,\ell \hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon}] \;\leq\; C_1 \exp(-C_0 \ell^{\,3\epsilon\kappa_1/4}). \tag{2.9}$$

Furthermore, with $\mathcal{B} = \mathcal{B}(\ell \hat{e}_1, 1) = \{x \in \mathbb{R}^d : |x - \ell \hat{e}_1| \leq 1\}$, for (possibly different) C_0 and C_1,

$$P[\exists\; b \in \mathcal{B} \textit{ with } d_{max}\,(M(0, b), \overline{0\,b}) \geq |b|^{\frac{3}{4}+\epsilon}] \;\leq\; C_1 \exp(-C_0 \ell^{\,3\epsilon\kappa_1/4}). \tag{2.10}$$

Proof. We will prove that, for some C_0 and C_1,

$$P[\exists\; b \in \mathcal{B} \text{ with } d_{max}\,(M(0, b), \overline{0\,\ell \hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon}] \;\leq\; C_1 \exp(-C_0 \ell^{\,3\epsilon\kappa_1/4}), \tag{2.11}$$

from which (2.9) follows immediately and (2.10) follows (for possibly different C_0 and C_1) from the facts that $|\,|b| - \ell| \leq 1$ and $|d_{max}\,(M(0, b), \overline{0\,b}) - d_{max}\,(M(0, b), \overline{0\,\ell \hat{e}_1})| \leq 1$.

We begin with the observations that

$$|T(u, v) - T(u, w)| \;\leq\; |q(v) - q(w)|^\alpha \quad \text{for all } u, v, w \in \mathbb{R}^d, \tag{2.12}$$

and that, for $q \in Q$ and $w \in \mathbb{R}^d$, $|q - q(w)| \leq 2|q - w|$, so also

$$|T(u, q) - T(u, w)| \;\leq\; (2|q - w|)^\alpha \quad \text{for all } q \in Q \text{ and } u, w \in \mathbb{R}^d. \tag{2.13}$$

Furthermore, repeated application of the triangle inequality to (2.12) gives that

$$|T(u,v) - T(u,w)| \;\leq\; (2|q(v) - v| + 2|v-w|)^{\alpha} \quad \text{for all } u,v,w \in \mathbb{R}^d. \tag{2.14}$$

Now let

$$\begin{aligned} A'_\ell &= \{x \in \mathbb{R}^d : \mathrm{Dist}\,(x, \overline{0\,\ell\hat{e}_1}) < \ell^{\frac{3}{4}+\epsilon}\}, \\ A_\ell &= \{x \in \mathbb{R}^d \setminus A'_\ell : \mathrm{Dist}\,(x, A'_\ell) < \ell^{\frac{3}{4}}\}, \text{ and} \\ A^+_\ell &= \{x \in \mathbb{R}^d \setminus A'_\ell : \mathrm{Dist}\,(x, A'_\ell) < \ell^{\frac{3}{4}} + \sqrt{d}\}. \end{aligned}$$

Additionally, let F_ℓ denote the event that $q(0), q(\ell\hat{e}_1) \in A'_\ell$ and every geodesic segment $\overline{q_k q_{k+1}}$ with either $|q_k| \leq \ell$ or $|q_{k+1}| \leq \ell$ has $|q_k - q_{k+1}| \leq \ell^{3/4}$. By an application of Lemma 5.2 (see (5.5)), F_ℓ satisfies $P[F^c_\ell] \leq C_1 \exp(-C_0 \ell^{3d/4})$. Furthermore, for large ℓ, on F_ℓ, for $b \in \mathcal{B}$ we have

$$\begin{aligned} d_{\max}\,(M(0,b), \overline{0\,\ell\hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon} &\implies \exists\; q \in Q \cap A_\ell \text{ on } M(0,b) \\ &\implies \exists\; q \in Q \cap A_\ell \text{ with } T(0,q) + T(q,b) = T(0,b) \\ &\implies \exists\; w \in \mathbb{Z}^d \cap A^+_\ell \text{ with } T(0,w) + T(w,\ell\hat{e}_1) \leq \\ &\qquad T(0,\ell\hat{e}_1) + 2((2\sqrt{d})^{\alpha} + (2|q(\ell\hat{e}_1) - \ell\hat{e}_1| + 2)^{\alpha}). \end{aligned}$$

This latter implication uses (2.13) and (2.14). It follows that, on $F_\ell \cap \{|q(\ell\hat{e}_1) - \ell\hat{e}_1| < \ell^{1/(2\alpha)}\}$, for large ℓ and $b \in \mathcal{B}$,

$$\begin{aligned} d_{\max}\,(M(0,b), \overline{0\,\ell\hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon} &\implies \exists\; w \in \mathbb{Z}^d \cap A^+_\ell \text{ with} \\ &\qquad T(0,w) + T(w,\ell\hat{e}_1) \leq T(0,\ell\hat{e}_1) + \ell^{\frac{1}{2}+\epsilon}. \end{aligned}$$

Hence

$$\begin{aligned} P[\exists\; b \in \mathcal{B} &\text{ with } d_{\max}\,(M(0,b), \overline{0\,\ell\hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon}] \\ &\leq C_1 \exp(-C_0 \ell^{3d/4}) \;+\; C_1 \exp(-C_0 \ell^{d/(2\alpha)}) \\ &\quad + \sum_{w \in \mathbb{Z}^d \cap A^+_\ell} P[T(0,w) + T(w,\ell\hat{e}_1) \leq T(0,\ell\hat{e}_1) + \ell^{\frac{1}{2}+\epsilon}]. \end{aligned} \tag{2.15}$$

The proof is completed by combining (2.7) and (2.15) with some elementary geometry.

Given $x, y \in \mathbb{R}^d$, let $\Delta(x,y) = \mu(|y| + |x-y| - |x|)$. Then on $\{T(0,w) + T(w,\ell\hat{e}_1) \leq T(0,\ell\hat{e}_1) + \ell^{\frac{1}{2}+\epsilon}\}$ at least one of $\big|T(0,w) - \mu|w|\big|$, $\big|T(w,\ell\hat{e}_1) - \mu|\ell\hat{e}_1 - w|\big|$, or $\big|T(0,\ell\hat{e}_1) - \mu\ell\big|$ must exceed $\tilde{\Delta}(\ell\hat{e}_1, w) \equiv (\Delta(\ell\hat{e}_1, w) - \ell^{\frac{1}{2}+\epsilon})/3$, so

$$\begin{aligned} P[T(0,w) + T(w,\ell\hat{e}_1) \leq T(0,\ell\hat{e}_1) + \ell^{\frac{1}{2}+\epsilon}] \;&\leq P[\big|T(0,w) - \mu|w|\big| > \tilde{\Delta}(\ell\hat{e}_1, w)] \\ &\quad + \; P[\big|T(w,\ell\hat{e}_1) - \mu|\ell\hat{e}_1 - w|\big| > \tilde{\Delta}(\ell\hat{e}_1, w)] \\ &\quad + \; P[\big|T(0,\ell\hat{e}_1) - \mu\ell\big| > \tilde{\Delta}(\ell\hat{e}_1, w)] \end{aligned} \tag{2.16}$$

We note that, for $w \in A_\ell^+$, $\Delta(\ell\hat{e}_1, w)$, and hence $\tilde{\Delta}(\ell\hat{e}_1, w)$, is at least of order $\ell^{\frac{1}{2}+2\epsilon}$ and at most of order $\ell^{\frac{3}{4}+\epsilon}$ as $\ell \to \infty$. For example, for $w = \ell\hat{e}_1/2 + (\ell^{\frac{3}{4}+\epsilon} + \sqrt{d})\hat{e}_2$, $\Delta(\ell\hat{e}_1, w)/\mu = 2((\ell/2)^2 + (\ell^{\frac{3}{4}+\epsilon} + \sqrt{d})^2)^{1/2} - \ell$ which is of order $\ell^{\frac{1}{2}+2\epsilon}$ by the Pythagorean theorem, while, for $w = (-\ell^{\frac{3}{4}+\epsilon} - \sqrt{d})\hat{e}_1$, $\Delta(\ell\hat{e}_1, w) = 2\mu(\ell^{\frac{3}{4}+\epsilon} + \sqrt{d}) = O(\ell^{\frac{3}{4}+\epsilon})$.

Each of the three probabilities in (2.16) can be expressed (using Euclidean invariance) in the form of the probability of (2.7) with ℓ replaced by some ℓ' between order $\ell^{\frac{3}{4}+\epsilon}$ and order ℓ, and with λ between order $\ell^{\frac{1}{2}+2\epsilon}$ and order $\ell^{\frac{3}{4}+\epsilon}$. We can bound these probabilities by replacing λ by the smaller $\lambda' = (\ell')^{\frac{1}{2}+\epsilon}$ so that the condition $(\ell')^{\frac{1}{2}+\epsilon} \leq \lambda' \leq (\ell')^{\frac{1}{2}+\kappa_2-\epsilon}$ is satisfied. Since A_ℓ can be contained in a Euclidean ball of radius order ℓ, we have

$$\begin{aligned} &P[\exists\, b \in \mathcal{B} \text{ with } d_{\max}(M(0,b), \overline{0\,\ell\hat{e}_1}) \geq \ell^{\frac{3}{4}+\epsilon}] \\ &\qquad \leq \; C_1 \exp(-C_0\ell^{3d/4}) + \; C_1 \exp(-C_0\ell^{d/(2\alpha)}) + \; C_1\ell^d \sup\{\exp(-C_0((\ell')^{\frac{1}{2}+\epsilon}/\sqrt{\ell'})^{\kappa_1})\}, \end{aligned} \tag{2.17}$$

where the supremum is over all ℓ' with $\ell^{\frac{3}{4}+\epsilon} \leq \ell' \leq \ell$. This yields (2.11) by taking $\ell' = \ell^{\frac{3}{4}+\epsilon}$ and noting that for large ℓ, $(\ell')^{\epsilon\kappa_1} \geq \ell^{(\frac{3}{4}+\epsilon)\epsilon\kappa_1} \geq \ell^{3\epsilon\kappa_1/4}$. ■

Theorem 2.4 immediately yields:

Corollary 2.5. *For $d \geq 2$, $\alpha > 1$ and any $\epsilon > 0$, the number N_ϵ of $q' \in Q$ such that $d_{\max}(M(0,q'), \overline{0\,q'}) \geq |q'|^{\frac{3}{4}+\epsilon}$ is a.s. finite.*

Proof. It follows from (2.10) of Theorem 2.4, rotational invariance, and an application of the Borel-Cantelli Lemma, that a.s. the events

$$\{\exists\ b \in \mathcal{B}(w,1) \text{ with } d_{max}\,(M(0,b), \overline{0\,b}) \geq |b|^{\frac{3}{4}+\epsilon}\}$$

occur for only finitely many $w \in (2/\sqrt{d})\mathbb{Z}^d$. The corollary follows since the $\mathcal{B}(w,1)$ cover $\mathbb{R}^d$. ∎

The next theorem, which itself is a consequence of this corollary, gives a different version of the inequality $\xi \leq 3/4$. To formulate the theorem, we need some notation. Let $C(x,\epsilon)$ for nonzero $x \in \mathbb{R}^d$ and $\epsilon \in [0,\pi)$ denote the cone

$$C(x,\epsilon) \;\equiv\; \{y \in \mathbb{R}^d : \theta(x,y) \leq \epsilon\}, \tag{2.18}$$

where θ is the angle (in $[0,\pi]$) between x and y. Recalling the definition of the spanning tree $R(q) = R_\alpha(q)$ formed by all geodesics $M(q,q')$ from q as given in Proposition 1.2, we define $R^{\mathrm{out}}(q,q')$ for $q' \in Q$ to be the set of all $q'' \in Q$ such that $M(q,q'')$ touches q' — i.e., it is the part of $R(q)$ going "outward" from q'. The next theorem states that for any q and all but finitely many q' (the number depending on q), any geodesic continuation of $M(q,q')$ must remain inside $q + C(q'-q, f^*(|q'-q|))$ where $f^*(\ell) \equiv \ell^{\frac{3}{4}+\epsilon}/\ell$. This was announced as Theorem 2 of [HoN2].

Theorem 2.6. *Let $d \geq 2$, $\alpha > 1$, $\epsilon \in (0,\frac{1}{4})$, and $f^*(\ell) = \ell^{-\frac{1}{4}+\epsilon}$. Then almost surely, for every $q \in Q$, for all but finitely many $q' \in Q$,*

$$R^{\mathrm{out}}(q,q') \;\subset\; q + C(q'-q, f^*(|q'-q|)). \tag{2.19}$$

Proof. It suffices to restrict attention to $q = q(0)$. From Lemma 5.2 (see (5.5)) and the Borel-Cantelli Lemma, it follows that there is some finite (random) L_0 so that for any geodesic segment $\overline{q_k\, q_{k+1}}$ with $|q_k| \geq L_0$, $|q_{k+1} - q_k| \leq |q_k|^{3/4}$. Theorem 2.6 is then a consequence of Corollary 2.5 and the following deterministic lemma.

Lemma 2.7. *Let $d \geq 2$ and $\delta \in (0, \frac{1}{4})$. Suppose $(q_i) = (q_1, q_2, \dots)$ is any sequence of distinct points in $\mathbb{R}^d$ with $|q_i| \to \infty$ such that for all large j,*

$$|q_{j+1} - q_j| \leq |q_j|^{3/4} \quad \text{and} \quad \mathit{Dist}(q_k, \overline{0\,q_j}) \leq |q_j|^{1-\delta} \quad \text{for } 1 \leq k < j. \tag{2.20}$$

Then there exists C_1 and $k^* > 0$ such that

$$\theta(q_k, q_j) \;\leq\; C_1 |q_k|^{-\delta} \quad \text{whenever } k^* \leq k < j. \tag{2.21}$$

Proof. Choose L large enough that $L^{3/4} < L^{1-\delta} < L/3$, and then choose k^* so that (2.20) holds and $|q_j| \geq L$ whenever $j \geq k^*$. Now suppose $k^* \leq k < j$.

Case 1: $|q_j| \leq 3|q_k|$. First note that we must have $\theta(q_k, q_j) < \pi/2$, for otherwise

$$\mathrm{Dist}\,(q_k, \overline{0\,q_j}) \;=\; |q_k| \;\geq\; \frac{|q_j|}{3} \;>\; |q_j|^{1-\delta},$$

which violates the second part of (2.20). It follows then from elementary geometric considerations that

$$\sin\theta(q_k, q_j) \;\leq\; \frac{|q_j|^{1-\delta}}{|q_k|} \;\leq\; 3^{1-\delta} |q_k|^{-\delta}.$$

Using that $\theta \leq \frac{\pi}{2} \sin\theta$ on $[0, \frac{\pi}{2})$, we see that $\theta(q_k, q_j) \leq C_1 |q_k|^{-\delta}$.

Case 2: $|q_j| > 3|q_k|$. We will construct a subsequence $(q_{i_0}, \dots, q_{i_n})$ of $(q_k, \dots q_j)$ such that: $q_{i_0} = q_k$; $q_{i_n} = q_j$; the $|q_{i_m}|$ are increasing; $|q_{i_{m+1}}| \leq 3|q_{i_m}|$ for $m+1 \leq n$; and, for $m+1 \leq n-1$, $|q_{i_{m+1}}| \geq 2|q_{i_m}|$. As we shall presently see, this is possible because, by the first part of (2.20), the sequence (q_i) stretches from q_k to q_j without any (relatively) large gaps. We then will have $|q_{i_m}| \geq 2^{m-1}|q_{i_0}| = 2^{m-1}|q_k|$ for $0 \leq m \leq n$, with the exponent $m-1$ (instead of m) to accomodate the case $m = n$. It follows from this and a repeated application of Case 1 that

$$\begin{aligned} \theta(q_k, q_j) \;=\; \theta(q_{i_0}, q_{i_n}) \;&\leq\; \sum_{m=0}^{n-1} \theta(q_{i_m}, q_{i_{m+1}}) \;\leq\; \sum_{m=0}^{n-1} C_1 |q_{i_m}|^{-\delta} \\ &\leq\; C_1 \left(\sum_{m=0}^{n-1} 2^{-(m-1)\delta} \right) |q_k|^{-\delta} \;\leq\; C_1 |q_k|^{-\delta}, \end{aligned}$$

where the final inequality holds for a larger C_1.

To construct the requisite (q_{i_m}), put $i_0 = k$ and suppose i_m has been selected. If $i_m = j$, put $n = m$ and stop. Otherwise, let $i_{m+1} = \max\{i : i_m < i \le j,\ |q_i| \le 3|q_{i_m}|\}$. By construction, the $|q_{i_m}|$ are increasing with $|q_{i_{m+1}}| \le 3|q_{i_m}|$. Furthermore, for $m+1 \le n-1$ (so also $i_{m+1} < j$), we must have $2|q_{i_m}| \le |q_{i_{m+1}}|$, for otherwise

$$\begin{aligned}|q_{i_{m+1}+1} - q_{i_{m+1}}| \ &\ge\ |q_{i_{m+1}+1}| - |q_{i_{m+1}}| \\ &>\ 3|q_{i_m}| - 2|q_{i_m}| \ =\ |q_{i_m}| \ \ge\ \frac{|q_{i_{m+1}}|}{3} \ \ge\ |q_{i_{m+1}}|^{3/4},\end{aligned}$$

in contradiction to the first part of (2.20). ■

2.3 Proof of Theorems 1.8-1.11. Suppose R is a tree whose vertex set is an infinite countable subset of $\mathbb{R}^d$ with u and u' two vertices of R. We define $R^{\text{out}}(u, u')$, as in the last subsection, to be the set of vertices u'' of R such that the (unique) path in R from u to u'' touches u'.

Definition. *For f a positive function on $(0,\infty)$, we say that such a tree R is f-straight at the vertex u if for all but finitely many vertices u' of R,*

$$R^{\text{out}}(u, u') \ \subset\ u + C(u' - u, f(|u' - u|)). \tag{2.22}$$

Theorem 2.6 is the statement that a.s., for every $q \in Q$, $R(q)$ is f^*-straight for $f^*(\ell) = \ell^{-\frac{1}{4}+\epsilon}$.

Definition. *Q', a subset of $\mathbb{R}^d$, is said to be asymptotically omnidirectional if for all finite K, $\{q/|q| : q \in Q'$ and $|q| > K\}$ is dense in S^{d-1}.*

Proposition 2.8. *Suppose R is a tree whose vertex set $U \subset \mathbb{R}^d$ is locally finite but asymptotically omnidirectional and such that every vertex has finite degree. Suppose further that for some $u \in U$, R is f-straight at u, where $f(\ell) \to 0$ as $\ell \to \infty$. Then R satisfies the following properties: (i) every semi-infinite path in R starting from u has an asymptotic direction; (ii) for every $\hat{x} \in S^{d-1}$, there is at least one semi-infinite path in R starting*

from u with asymptotic direction $\hat{x}$; (iii) the set $V(u)$ of $\hat{x}$'s such that there is more than one semi-infinite path starting from u with asymptotic direction $\hat{x}$ is dense in S^{d-1}.

Proof. Let $u = u_1, u_2, \dots$ be a semi-infinite path in R. Then f-straightness implies that for large m, the angle $\theta(u_n - u, u_m - u) \leq f(|u_m - u|)$ for $n \geq m$. Since $|u_m| \to \infty$ as $m \to \infty$ (because U is locally finite), it follows that $u_n/|u_n|$ converges, proving (i). Since U is asymptotically omnidirectional and each vertex has finite degree, it follows that starting from $v_1 = u$, one can for a given $\hat{x}$ inductively construct a semi-infinite path $v_1, v_2, \dots$ in R such that for each j, $R_{\text{out}}(u, v_j)$ contains a sequence (depending on j) $u_1, u_2, \dots$ with $u_n/|u_n| \to \hat{x}$. But (i) shows that $v_j/|v_j|$ tends to some $\hat{y}$ and then f-straightness implies $\theta(\hat{x}, v_j - u) \leq f(|v_j - u|)$ so that letting $j \to \infty$ yields $\hat{x} = \hat{y}$, proving (ii).

Given any (large) finite K, one can consider those (finitely many) vertices v with $|v| > K$ such that no other vertex w on the path from u to v has $|w| > K$. By taking a subset of these v's, one obtains a finite set of vertices $v_1^{(K)}, \dots, v_{m(K)}^{(K)}$ with $|v_j^{(K)}| > K$ such that the $R^{\text{out}}(u, v_j^{(K)})$'s are disjoint and their union includes all but finitely many vertices of U. For a given K, let G_j denote the set of $\hat{x}$'s such that some semi-infinite path from u passing through $v_j^{(K)}$ has asymptotic direction $\hat{x}$. Then by (ii), $\cup_j G_j = S^{d-1}$. On the other hand, by f-straightness, each G_j is a subset of the (small) spherical cap $\{\hat{x} : \theta(\hat{x}, v_j^{(K)}) \leq f(|v_j^{(K)} - u|) \leq \epsilon(K)\}$ where $\epsilon(K) \to 0$ as $K \to \infty$ (since $|v_j^{(K)}| > K$). Furthermore, by the same arguments that proved (ii), each G_j is a *closed* subset of S^{d-1}. It follows that $V(u)$ contains, for each K, $\cup_{j \leq m(K)} \partial G_j$, where ∂G_j denotes the usual boundary (G_j less its interior). Since $\epsilon(K) \to 0$ as $K \to \infty$, we obtain (iii) by standard arguments. ■

Proof of Theorems 1.8, 1.9 and 1.10. These three theorems are all essentially immediate consequences of Proposition 2.8 and the (easily proven) fact that Q is a.s. locally finite and asymptotically omnidirectional.

Proof of Theorem 1.11. The only part of Theorem 1.11 that remains to be proved (i.e., that does not immediately follow from Theorems 1.6 and 1.8) is that $(\hat{x}, \hat{y})$-geodesics with $\hat{y} \neq -\hat{x}$ do not exist, even for nondeterministic $\hat{x}$ and $\hat{y}$ depending on Q. To prove this, it suffices to show, for each $\delta > 0$, that this is the case with the further restrictions that $\theta(\hat{y}, -\hat{x}) > \delta$ and that the $(\hat{x}, \hat{y})$-geodesic touches $q(0)$. Let E_k denote the event that there exist $q, q'' \in \mathbb{R}^d$ with $|q'|, |q''| \in [k, k + k^{3/4}]$, $\theta(q'', -q') > \delta/2$, and with $M(q', q'')$ touching $q(0)$. By arguments like those in the proofs of Theorem 2.3 and Corollary 2.4 one can prove that $P[E_k$ infinitely often$] = 0$ and that this leads to the non-existence of $(\hat{x}, \hat{y})$-geodesics passing through $q(0)$ with $\theta(\hat{y}, -\hat{x}) > \delta$. We leave further details to the reader. ■

3. Proof of Theorem 2.1

In many respects, our proof of Theorem 2.1 parallels the arguments in [Ke2], where analogous results for lattice FPP are presented. However, our Euclidean framework presents a host of technical issues. For such technical reasons we will need to work with certain approximations of T_ℓ. With $\bar{Q}$ any locally finite subset of $\mathbb{R}^d$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ any continuous strictly increasing convex function with $\phi(0) = 0$ (the cost function) and a and b arbitrary, and possibly random, points in $\mathbb{R}^d$, define

$$T[\bar{Q}, \phi, a, b] \;=\; \inf\left\{ \sum_{j=1}^{k-1} \phi(|q_j - q_{j+1}|) : k \geq 2, q_1 = a, q_k = b, q_j \in \bar{Q} \text{ for } 1 < j < k \right\}. \tag{3.1}$$

So, for example, with $\phi_\infty(t) \equiv t^\alpha$, we have $T_\ell = T[Q, \phi_\infty, q(0), q(\ell \hat{e}_1)]$. Our first approximation to T_ℓ, denoted by T'_ℓ, is defined by

$$T'_\ell \;=\; T[Q, \phi_\infty, 0, \ell \hat{e}_1].$$

It would seem that T'_ℓ is a more natural quantity to study, since the paths under consideration actually start at 0 and end at $\ell \hat{e}_1$. Unfortunately, T'_ℓ does not obey a triangle inequality whereas T_ℓ does. For our second approximation, T''_ℓ, we will need a collection

of subsets $Q_\ell \subset Q$ to be defined later (see above (3.6)) and a family of cost functions ϕ_ℓ defined as

$$\phi_\ell(t) \;=\; \begin{cases} t^\alpha & \text{if } t \le h_\ell \\ h_\ell^\alpha + \alpha h_\ell^{\alpha-1}(t-h_\ell) & \text{otherwise,} \end{cases} \tag{3.2}$$

where $h_\ell = \max(h_0, h_1\ell^{\kappa_3})$ with $\kappa_3 = 1/(2\alpha)$, and with both $h_0 \ge 1$ and $h_1 \ge h_0$ to be specified later (see above (3.16) and below (3.28)). Note that $\phi_\ell(t) \uparrow \phi_\infty(t) = t^\alpha$ as $\ell \to \infty$; we will also have $Q_\ell \uparrow Q$. We now define

$$T''_\ell \;=\; T[Q_\ell, \phi_\ell, 0, \ell\hat{e}_1].$$

These approximations to Q and ϕ_∞ will play a role similar to a truncation argument allowing $T''_\ell - ET''_\ell$ to be expressed as the limit of a martingale with bounded differences.

Throughout this section, we use the following notation. We let

$$q(0) = r_1, r_2, \ldots, r_K = q(\ell\hat{e}_1),$$
$$0 = r'_1, r'_2, \ldots, r'_{K'} = \ell\hat{e}_1, \text{ and}$$
$$0 = r''_1, r''_2, \ldots, r''_{K''} = \ell\hat{e}_1$$

achieve the infima in (3.1) corresponding to T_ℓ, T'_ℓ, and T''_ℓ respectively. We use L_k to denote the "link" (i.e., straight line segment) $\overline{r_k r_{k+1}}$, and we use $\overline{r}$ to denote the polygonal path $L_1 L_2 \ldots L_{K-1}$ with analogous interpretations of L'_k, L''_k, $\overline{r'}$, and $\overline{r''}$. For any link L, $|L|$ will be its Euclidean length. Also, for any cost function ϕ of the form (3.2) and $a, b \in \mathbb{R}^d$, let

$$\mathcal{W}_\phi(a,b) \;=\; \{c \in \mathbb{R}^d : \phi(|a-c|) + \phi(|c-b|) \le \phi(|a-b|)\}$$

and put $\mathcal{W}(a,b) = \mathcal{W}_{\phi_\infty}(a,b)$. A number of properties of these subsets of $\mathbb{R}^d$ are gathered in Lemma 5.1 of Section 5 below and used in the proof of the next lemma.

With an appropriate Q_ℓ and h_ℓ the random variables T_ℓ, T'_ℓ, and T''_ℓ are related as follows:

Lemma 3.1. *With $\kappa_4 = d/\alpha$ and for some constants C_0 and C_1,*

$$P[|T_\ell - T'_\ell| > x] \ \le\ C_1 \exp(-C_0 x^{\kappa_4}), \ \textit{for } x > 0 \tag{3.3}$$

and

$$P[T'_\ell \neq T''_\ell] \ \le\ C_1 \exp(-C_0 \ell^{\kappa_3}). \tag{3.4}$$

Proof of (3.3). The left side of (3.4) is ill-defined until the Q_ℓ and h_ℓ are chosen; we defer its proof. This does not apply to inequality (3.3), which is easier to prove. Let $\Gamma(a) = \sup\{|c-a| : \mathcal{W}(a,c) \cap Q = \emptyset\}$, and set $\Gamma_\ell = \Gamma(0) + \Gamma(\ell\hat{e}_1)$. Then

$$T'_\ell \ \le\ T_\ell + |q(0)|^\alpha + |q(\ell\hat{e}_1) - \ell\hat{e}_1|^\alpha \ \le\ T_\ell + \Gamma_\ell^\alpha,$$

and, similarly, on $\{K' \ge 3\}$,

$$T_\ell \ \le\ T'_\ell + |q(0) - r'_2|^\alpha + |q(\ell\hat{e}_1) - r'_{K'-1}|^\alpha \ \le\ T'_\ell + 2^\alpha \Gamma_\ell^\alpha,$$

while on $\{K' = 2\}$, $\Gamma(0) \ge \ell$ so

$$T_\ell \ \le\ (|q(0)| + \ell + |q(\ell\hat{e}_1) - \ell\hat{e}_1|)^\alpha \ \le\ (\Gamma(0) + \Gamma(0) + \Gamma(\ell\hat{e}_1))^\alpha \ \le\ 2^\alpha\Gamma_\ell^\alpha \ \le\ T'_\ell + 2^\alpha\Gamma_\ell^\alpha.$$

Collectively these bounds yields $|T_\ell - T'_\ell| \le 2^\alpha \Gamma_\ell^\alpha$. We complete the proof of (3.3) by using the Remark following Lemma 5.2 below (see (5.4)) to conclude that, for appropriate C_0 and C_1,

$$P[2^\alpha\Gamma_\ell^\alpha > x] \ \le\ P[\Gamma(0) > x^{1/\alpha}/4] + P[\Gamma(\ell\hat{e}_1) > x^{1/\alpha}/4] \ \le\ C_1\exp(-C_0 x^{d/\alpha}). \tag{3.5}$$

Our proof of (2.4) divides into the two cases $0 \le \ell \le 1$ and $\ell > 1$. We are really only interested in the second (much more difficult) case, but proving the first case illustrates the sort of technical difficulties created by our definition of T_ℓ. We have:

Lemma 3.2. *For some constant C_1, $Var T_\ell \le C_1 \ell$ whenever $\ell \le 1$.*

Proof. If we were working with T'_ℓ instead of T_ℓ, this case would be straightforward since, for $\ell \le 1$, $(T'_\ell)^2 \le \ell^{2\alpha} \le \ell$. On the other hand, although $T_\ell = 0$ for ℓ small enough that $q(0) = q(\ell \hat{e}_1)$, no matter how small ℓ is, among those Poisson particle configurations for which $q(0) \ne q(\ell\hat{e}_1)$, $|q(0) - q(\ell\hat{e}_1)|$ (and T_ℓ) can be arbitrarily large. Looking a little closer, for any fixed $\ell \le 1$ let $\tilde{D}$ denote the event $\{q(0) \ne q(\ell\hat{e}_1)\}$. For $\rho > 0$, on $\{|q(0)| = \rho\}$ we have

$$T_\ell^2 \ \le\ |q(0) - q(\ell\hat{e}_1)|^{2\alpha} I_{\tilde{D}} \ \le\ (|q(0)| + \ell + |q(\ell\hat{e}_1)|)^{2\alpha} I_{\tilde{D}} \ \le\ (2\rho + 2)^{2\alpha} I_{\tilde{D}},$$

where $I_{\tilde{D}}$ denotes the indicator of the event $\tilde{D}$. Letting A_ρ denote the event that there is a particle in the annulus $\{x \in \mathbb{R}^d : \rho < |x| < \rho + 2\ell\}$, we have

$$\{|q(0)| = \rho\} \cap \tilde{D} \ \subset\ \{|q(0)| = \rho\} \cap A_\rho,$$

so

$$E[T_\ell^2 \big| |q(0)| = \rho] \ \le\ (2\rho+2)^{2\alpha} P[A_\rho \big| |q(0)| = \rho] \ \le\ C_1 (2\rho+2)^{2\alpha} (\rho+2)^{d-1} \ell,$$

and

$$\begin{aligned} \operatorname{Var} T_\ell \ \le\ ET_\ell^2 \ &=\ \int_{\rho \ge 0} E[T_\ell^2 \big| |q(0)| = \rho] dP[|q(0)| \le \rho] \\ &\le\ \ell C_1 \int_{\rho \ge 0} (2\rho+2)^{2\alpha+d-1} dP[|Q(0)| \le \rho] \ =\ \ell C_1. \end{aligned}$$

(Recall that according to our conventions, the two instances of C_1 in the preceding equation represent different constants.) ■

Proceeding with the case $\ell > 1$, we define:

$$S''_\ell \ =\ \sum_{j=1}^{K''-1} \phi_\ell^2(|L''_j|).$$

We do the case $\ell > 1$ in three steps.

Step 1: For any $\ell > 0$, $\operatorname{Var} T''_\ell \;\leq\; 2^{2\alpha+1} E S''_\ell$. We note that this inequality is also ill-defined until the Q_ℓ and h_ℓ are specified. We presently define the Q_ℓ; it turns out that Step 1 holds for any h_ℓ. Throughout this paper, for any length $\eta > 0$, the "η-boxes" will refer to the interior-disjoint d-dimensional cubes whose vertices collectively are $\eta \cdot (\mathbb{Z}^d + (\frac{1}{2}, \ldots, \frac{1}{2}))$. For any η, the η-boxes may be associated with the $\mathbb{Z}^d$ lattice in the natural way: $\nu \in \mathbb{Z}^d$ is associated with the box centered at $\eta\nu$. Two η-boxes are called adjacent if they share a common $(d-1)$-dimensional face (i.e., if their associated sites in $\mathbb{Z}^d$ are nearest neighbors). For any Borel subset $S \subset \mathbb{R}^d$, let $\mathcal{F}(S)$ denote the σ-subfield of $\mathcal{F}$ generated by all events of the form $\{\omega : Q(\omega) \cap B \neq \emptyset\}$ where B ranges over all Borel subsets of S. Clearly we may (and do) replace $\mathcal{F}$ with the possibly smaller $\mathcal{F}(\mathbb{R}^d)$. Now fix any $\ell > 0$ and let $(B_m : m = 1, 2, \ldots)$ denote the $(\epsilon/3^{\lfloor \ell \rfloor})$-boxes ($\epsilon$ is a quantity that depends only on d and is specified in Step 2 below) enumerated in some order. We note that, in general, if η' is an odd integral multiple of η then the η-boxes are nested in the η'-boxes so, in particular, the $(\epsilon/3^{\lfloor \ell \rfloor})$-boxes are nested in the ϵ-boxes. Let q_m denote the leftmost particle in $Q \cap B_m$ (provided such a particle exists) and let $Q_\ell = \{q_m\} \subset Q$ denote the set of all such leftmost particles.

Let $\mathcal{F}_m = \mathcal{F}(B_1 \cup \cdots \cup B_m)$ with $\mathcal{F}_0 = \{\emptyset, \Omega\}$, so $\mathcal{F}_m \uparrow \mathcal{F}$ as $m \to \infty$. Set

$$\Delta_m \;=\; E(T''_\ell | \mathcal{F}_m) - E(T''_\ell | \mathcal{F}_{m-1})$$

so that

$$T''_\ell - E T''_\ell \;=\; \sum_{m=1}^{\infty} \Delta_m, \text{ and } \operatorname{Var} T''_\ell \;=\; \sum_{m=1}^{\infty} E\Delta_m^2.$$

This holds since T''_ℓ is bounded by ℓ^α. Now set $\tilde{\mathcal{F}}_m = \mathcal{F}(\mathbb{R}^d \setminus B_m)$ and define

$$\tilde{\Delta}_m \;=\; T''_\ell - E(T''_\ell | \tilde{\mathcal{F}}_m).$$

Then we have that $E\Delta_m^2 \;\leq\; E\tilde{\Delta}_m^2$ since $\Delta_m = E(\tilde{\Delta}_m|\mathcal{F}_m)$ giving that

$$\operatorname{Var} T_\ell'' \;\leq\; \sum_{m=1}^{\infty} E\tilde{\Delta}_m^2 \tag{3.6}$$

In general, if X and Y are L^2 random variables with Y measurable with respect to some σ-field $\mathcal{G}$, then

$$E[(X - E[X|\mathcal{G}])^2|\mathcal{G}] \;\leq\; E[(X-Y)^2|\mathcal{G}]. \tag{3.7}$$

Put $T_\ell^{(m)} = T[Q_\ell \setminus B_m, \phi_\ell, 0, \ell\hat{e}_1]$; so $T_\ell^{(m)}$ is the minimal passage time from 0 to $\ell\hat{e}_1$ with respect to the ϕ_ℓ cost function using points in Q_ℓ other than in B_m, and $T_\ell^{(m)}$ is $\tilde{\mathcal{F}}_m$-measurable. Hence, with $U_m = (T_\ell^{(m)} - T_\ell'')^2$ we have

$$E[\tilde{\Delta}_m^2|\tilde{\mathcal{F}}_m] \;\leq\; E[U_m|\tilde{\mathcal{F}}_m], \text{ and } E\tilde{\Delta}_m^2 \;\leq\; EU_m. \tag{3.8}$$

Let $\bar{R}_m$ denote the event that q_m exists and equals r_i'' for some i. On the event $\bar{R}_m$, define the random variable $k(m)$ by the relation $r_{k(m)}'' = q_m$. (Off of $\bar{R}_m$, the value of $k(m)$ is irrelevant.) Then

$$0 \;\leq\; T_\ell^{(m)} - T_\ell'' \;\leq\; \phi_\ell(|r_{k(m)-1}'' - r_{k(m)+1}''|)I_{\bar{R}_m}\,,$$

so, using Lemma 5.3 in the second inequality below,

$$\begin{aligned} S_\ell \;:=\; \sum_{m=1}^{\infty} U_m \;&\leq\; \sum_m \phi_\ell^2(|r_{k(m)-1}'' - r_{k(m)+1}''|)I_{\bar{R}_m} \\ &=\; \sum_{k=2}^{K''-1} \phi_\ell^2(|r_{k-1}'' - r_{k+1}''|) \\ &\leq\; \sum_{k=2}^{K''-1} 2^{2\alpha}(\phi_\ell^2(|r_{k-1}'' - r_k''|) + \phi_\ell^2(|r_k'' - r_{k+1}''|)) \\ &\leq\; 2^{2\alpha+1}\sum_{k=1}^{K''-1} \phi_\ell^2(|r_k'' - r_{k+1}''|) \;=\; 2^{2\alpha+1}S_\ell''. \end{aligned} \tag{3.9}$$

Combining (3.6), (3.8), and (3.9) gives $\operatorname{Var} T_\ell'' \leq 2^{2\alpha+1}ES_\ell''$.

Step 2. For some constant C_1, $ES''_\ell \le C_1\ell$ (for $\ell > 1$). In fact, with $\kappa_5 = 1/(4\alpha+2)$ and for some constants C_0 and C_1,

$$P[S''_\ell > x] \ \le\ C_1 \exp(-C_0 x^{\kappa_5}) \text{ for all } x \ge C_1\ell. \tag{3.10}$$

As this step is the heart of the proof, we begin by describing the overall structure of the argument. A main ingredient is Lemma 3.3 below, which gives a large deviation bound for T''_ℓ, obtained by constructing a suboptimal path for the cost function ϕ_ℓ. Such arguments do not directly yield bounds such as (3.10) for S''_ℓ because the definition of S''_ℓ involves replacing ϕ_ℓ by ϕ_ℓ^2 while still using the links L''_j that are optimal for ϕ_ℓ. So we separate the links L''_j into short and long ones and correspondingly write $S''_\ell = S_1 + \tilde{S}$ (with $\tilde{S}$ further decomposed as $S_2 + S_3$). The tail of S_1 is directly estimated by that of T''_ℓ, but the analysis of $\tilde{S}$ requires more work. We will choose an appropriately small ϵ, relate the path r'' to a kind of path formed from ϵ-boxes and then control the tail of $\tilde{S}$ by a combination of percolation and lattice animal estimates for the path formed from ϵ-boxes. Now, to work.

We will call any finite sequence of distinct η-boxes an "η-box path" if the first box contains the origin and the boxes are sequentially adjacent; the path's "length" will refer to the number of boxes on the path. We call an η-box "occupied" if it contains a Poisson particle. Pick $0 < \epsilon \le 1$ small enough so that, (i) as in the proof of Lemma 3 of [HoN1], the events

$$F_x \ =\ \{\exists \text{ an } \epsilon\text{-box path of length } m \ge x \text{ with at least } m/2d \text{ occupied boxes}\}$$

satisfy $PF_x \le (1-e^{-1})^{-1}e^{-x}$, and (ii) $17\epsilon\sqrt{d}$ is strictly less than the critical radius R_c^* for continuum percolation (discussed just before Conjecture 1 in Subsection 1.2). The strict positivity of R_c^* can be shown by standard arguments — see, e.g., Theorem 3.2 of [MR]. (We remark that for any ℓ, by the construction of Q_ℓ an ϵ-box is occupied (by a Poisson particle in Q) if and only if it contains a particle in Q_ℓ.) Consider the ϵ-box path

$\beta = (\beta_1, \ldots, \beta_{\tilde{M}(\ell\hat{e}_1)})$ from 0 to $\ell\hat{e}_1$ constructed as follows: β_1 is the ϵ-box that contains 0; if $\overline{r''}$ does not end inside of β_k, β_{k+1} is the (a.s. adjacent) ϵ-box that $\overline{r''}$ enters when it last exits β_k. Here $\tilde{M}(\ell\hat{e}_1)$ is the random number of boxes along this box path. It follows as in the proof of Lemma 3 of [HoN1] that, for large x:

$$T''_\ell \;\geq\; \frac{\phi_\ell(\epsilon)x}{3d} \;=\; \frac{\epsilon^\alpha x}{3d} \text{ on } F_x^c \cap \{\tilde{M}(\ell\hat{e}_1) \geq x\},$$

(the equality above holds since $\epsilon \leq 1 \leq h_\ell$) and hence

$$P[\tilde{M}(\ell\hat{e}_1) \geq x] \;\leq\; PF_x + P\Big[T''_\ell \geq \frac{\epsilon^\alpha x}{3d}\Big] \;\leq\; (1 - e^{-1})\exp(-x) + P\Big[T''_\ell \geq \frac{\epsilon^\alpha x}{3d}\Big]. \quad (3.11)$$

The ϵ-box path β covers the midpoint of any sufficiently long link in $\overline{r''}$. To see this, let $\overline{ab} = \overline{r''_k r''_{k+1}}$ be any link in $\overline{r''}$ and let c be its midpoint. Suppose β_{i^*} is the last ϵ-box along β that touches either $\overline{ac}$ or any link that precedes $\overline{ab}$ on $\overline{r''}$. If $i^* = \tilde{M}(\ell\hat{e}_1)$, put $\rho = \beta_{i^*}$; otherwise put $\rho = \beta_{i^*} \cup \beta_{i^*+1}$. Then ρ touches c^* and c^{**} satisfying at least one of the following:

$$c^* \in \overline{ac} \text{ and } c^{**} \in \overline{cb}, \quad (3.12)$$

$$c^* \in L^* \text{ and } c^{**} \in \overline{cb} \text{ where } L^* \text{ is a link on } \overline{r''} \text{ preceding } \overline{ab}, \quad (3.13)$$

$$c^* \in \overline{ac} \text{ and } c^{**} \in L^{**} \text{ where } L^{**} \text{ is a link on } \overline{r''} \text{ succeeding } \overline{ab}, \quad (3.14)$$

$$c^* \in L^* \text{ and } c^{**} \in L^{**} \text{ with } L^* \text{ and } L^{**} \text{ as in (3.13) and (3.14)}. \quad (3.15)$$

Now (3.12) implies that $c \in \rho$. On the other hand, by the No Doubling Back Proposition of [Ho] (stated below as Lemma 5.5), (3.13) implies

$$\frac{1}{2}|a-b| \;=\; |a-c| \;\leq\; |a-c^{**}| \;\leq\; 16|c^* - c^{**}| \;\leq\; 16\epsilon\sqrt{d+3},$$

while (3.14) similarly implies

$$\frac{1}{2}|a-b| \;=\; |c-b| \;\leq\; |c^* - b| \;\leq\; 16|c^* - c^{**}| \;\leq\; 16\epsilon\sqrt{d+3},$$

and (3.15) implies

$$
\begin{aligned}
|a-b| &\le |\text{ending point of } L^* - \text{starting point of } L^{**}| \\
&\le 33|c^* - c^{**}| \le 33\epsilon\sqrt{d+3}.
\end{aligned}
$$

It follows that $c \in \rho$ provided $|a-b| > 33\epsilon\sqrt{d+3}$.

Choose λ to be an odd integral multiple of ϵ (so the ϵ-boxes are nested in the λ-boxes) with λ large enough that the probability that any fixed λ-box contains no Poisson particle (equivalently, no Q_ℓ particle) is below the critical probability for site percolation on the nearest neighbor $\mathbb{Z}^d$ lattice. If the midpoint of a link L''_k is touched by the ϵ-box path β, then a.s. it is touched by only one of the ϵ-boxes on β; let $\nu(L''_k)$ denote the λ-box that contains this ϵ-box. (If the midpoint of L''_k is not so touched, $\nu(L''_k)$ is undefined.) For any λ-box ν, let $|\mathcal{C}_\nu|$ denote the size (i.e., the cardinality) of the nearest-neighbor cluster $\mathcal{C}_\nu$ of unoccupied λ-boxes at ν. The quantity y_0 in (3.18) below will be specified later but depends only on d. We choose h_0 sufficiently large such that $|L''_k| > h_0$ implies:

$$\nu(L''_k) \text{ is defined;} \tag{3.16}$$

$$\text{if } L''_j \neq L''_k \text{ is another link with } |L''_j| > h_0 \text{ then } \nu(L''_j) \neq \nu(L''_k)\text{; and} \tag{3.17}$$

$$\nu(L''_k) \text{ is unoccupied, moreover } |\mathcal{C}_{\nu(L''_k)}| \ge y_0^{1/(2\alpha)}|L''_k|. \tag{3.18}$$

We can ensure (3.16) by the preceding discussion and (3.17) also follows easily for large h_0 from the No Doubling Back Proposition (Lemma 5.5). Since the interior of the region $\mathcal{W}_{\phi_\ell}(r''_k, r''_{k+1})$ contains no Q_ℓ particles, Lemma 5.4 furnishes (3.18) for h_0 sufficiently large (depending on y_0).

We split S''_ℓ into three pieces as follows:

$$\begin{aligned} S''_\ell &= S_1 + S_2 + S_3, \text{ where} \\ S_1 &= \sum_{k\,:\,|L''_k| \le h_0} \phi_\ell^2(|L''_k|), \\ S_2 &= I_{\{\tilde{M}(\ell\hat{e}_1) \ge x\}} \sum_{k\,:\,|L''_k| > h_0} \phi_\ell^2(|L''_k|), \text{ and} \\ S_3 &= I_{\{\tilde{M}(\ell\hat{e}_1) < x\}} \sum_{k\,:\,|L''_k| > h_0} \phi_\ell^2(|L''_k|). \end{aligned}$$

Now $S_1 \le h_0^\alpha T''_\ell$, so

$$P[S_1 > x] \le C_1 \exp(-C_0 x^{\kappa_1}) \text{ for all } x \ge C_1 \ell \tag{3.19}$$

will follow with $\kappa_1 = \min(1, d/\alpha)$ from:

Lemma 3.3. *There exist constants C_0 and C_1 such that, for $T = T_\ell$, $T = T'_\ell$, or $T = T''_\ell$, $P[T > x] \le C_1 \exp(-C_0 x^{\kappa_1})$ for $x \ge C_1 \ell$.*

Proof. We first prove (in detail) the case $T = T'_\ell$. For $a \in \mathbb{R}^d$ and $t \ge 0$, let

$$\begin{aligned} \mathcal{D}_t(a) &= \{a + b \in \mathbb{R}^d : 0 \le b_1 \le t; 0 \le \sigma_i b_i \le b_1 \text{ for } 2 \le i \le d\}, \text{ where} \\ \sigma_i &= -1 \text{ if } a_i \ge 0, \ 1 \text{ otherwise.} \end{aligned}$$

Then the d-dimensional volume of $\mathcal{D}_t(a)$ is $\int_0^t s^{d-1}\, ds = t^d/d$. Also, for $b \in \mathcal{D}_t(a)$ we have

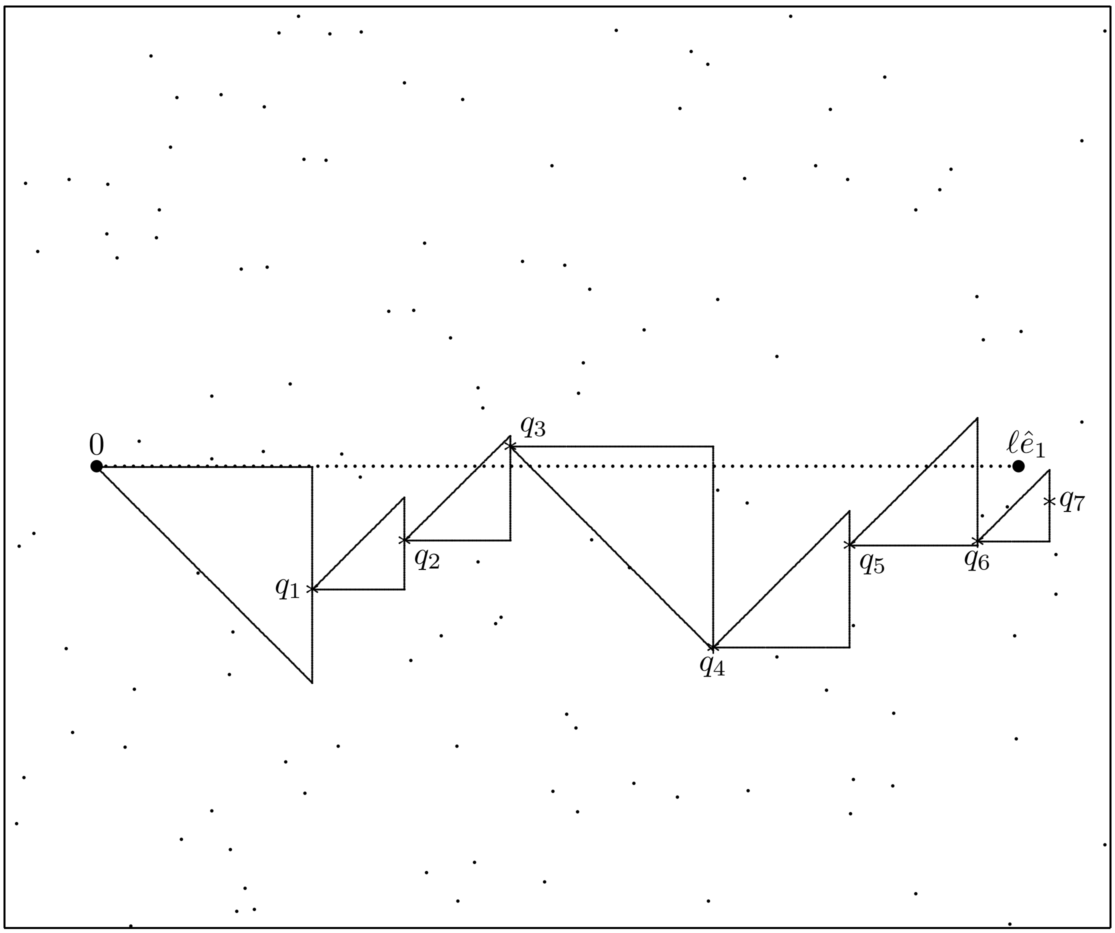
$$\max_{2 \le i \le d} |b_i| \le \max(t, \max_{2 \le i \le d} |a_i|). \tag{3.20}$$

Let $q_0 = 0$ and define q_n and R_n inductively for $n \ge 1$ (See Figure 1 for the picture when $d = 2$) by the relation

$$\tilde{R}_n = \inf\{t > 0 : \text{ there exists a Poisson particle } q_n \ne q_{n-1} \text{ in } \mathcal{D}_t(q_{n-1})\},$$

and let $\tilde{R}_n^* = \max_{1 \le m \le n} \tilde{R}_m$.

Figure 1.



Now $|q_{n-1} - q_n| \leq \tilde{R}_n\sqrt{d}$ and it follows from (3.20) that $|q_N - \ell\hat{e}_1| \leq \tilde{R}_N^*\sqrt{d}$ where

$$N = \min\{n : \tilde{R}_1 + \cdots + \tilde{R}_n \geq \ell\}.$$

Hence

$$\begin{aligned} T'_\ell &\leq |q_0 - q_1|^\alpha + \cdots + |q_{N-1} - q_N|^\alpha + |q_N - \ell\hat{e}_1|^\alpha \\ &\leq (\tilde{R}_1\sqrt{d})^\alpha + \cdots + (\tilde{R}_N\sqrt{d})^\alpha + (\tilde{R}_N^*\sqrt{d})^\alpha \\ &\leq 2d^{\alpha/2}(\tilde{R}_1^\alpha + \cdots + \tilde{R}_N^\alpha). \end{aligned}$$

It follows that for any $n > 0$

$$
\begin{aligned}
P[T'_\ell > x] \;&\leq\; P[2d^{\alpha/2}(\tilde{R}_1^\alpha + \cdots + \tilde{R}_n^\alpha) > x] \;+\; P[n < N] \\
&\leq\; P[2d^{\alpha/2}(\tilde{R}_1^\alpha + \cdots + \tilde{R}_n^\alpha) > x] \;+\; P[\tilde{R}_1 + \cdots + \tilde{R}_n < \ell]
\end{aligned}
\tag{3.21}
$$

Now the $\tilde{R}_i$, and hence the $\tilde{R}_i^\alpha$, are i.i.d., with $P[\tilde{R}_i^\alpha > r] = P[\tilde{R}_i > r^{1/\alpha}] = \exp(-\frac{1}{d} r^{d/\alpha})$. Taking $n = \lceil cx \rceil$ in (3.21), it follows from [N] that, for sufficiently small c, there exist C_0 and C_1 such that

$$
P[2d^{\alpha/2}(\tilde{R}_1^\alpha + \cdots + \tilde{R}_{\lceil cx \rceil}^\alpha) > x] \;\leq\; C_1 \exp(-C_0 x^{\kappa_1}) \text{ for all } x.
$$

Also, for this choice of c, it follows from elementary large deviation results for i.i.d. random variables (see, e.g., Sec. 1.9 of [D]) that, for possibly larger C_1 and smaller C_0, we have:

$$
P[\tilde{R}_1 + \cdots + \tilde{R}_{\lceil cx \rceil} < \ell] \;\leq\; C_1 \exp(-C_0 x) \text{ for } x \geq C_1 \ell.
$$

The lemma therefore follows for T'_ℓ.

This extends easily to $T = T_\ell$ (with the same exponent κ_1) by applying the first part of Lemma 3.1. To apply the T'_ℓ result to T''_ℓ, note that the fact that $\phi_\ell(t) \leq \phi_\infty(t) = t^\alpha$ is helpful, so the only difficulty is that the sequence of Poisson particles $q_1, \ldots, q_N$ constructed above are not necessarily in Q_ℓ. However, there is always a Q_ℓ particle $\tilde{q}_i$ within a distance $(\epsilon/3^\ell)\sqrt{d} \leq \sqrt{d}$ of each q_i constructed above. It is not hard to see that the sequence $(\tilde{q}_i)$ produces a path whose passage time has a distribution with the requisite tail, again with the same exponent κ_1. ■

We bound the tail of S_2 by the simple estimate

$$
\begin{aligned}
P[S_2 > x] \;&\leq\; P[\tilde{M}(\ell \hat{e}_1) > x] \\
&\leq\; (1 - e^{-1}) \exp(-x) + C_1 \exp\left(- C_0 \Big(\frac{\epsilon^\alpha x}{3d} \Big)^{\kappa_1} \right) \text{ for } \frac{\epsilon^\alpha x}{3d} \geq C_1 \ell \\
&\leq\; C_1 \exp(-C_0 x^{\kappa_1}) \text{ for all } x \geq C_1 \ell.
\end{aligned}
\tag{3.22}
$$

Here we use (3.11) and Lemma 3.3; the final inequality holds for possibly larger C_1 and smaller C_0 since $\kappa_1 \leq 1$.

Finally, we bound the tail of S_3. Let ξ'' denote the collection of λ-boxes that contain an ϵ-box on β. If $\Xi_0 = \{\text{all } \mathbb{Z}^d \text{ lattice animals containing the origin}\}$, then $\xi'' \in \Xi_0$ in the sense that the sites in $\mathbb{Z}^d$ associated with the boxes in ξ'' form a lattice animal containing the origin. Then, using (3.16), (3.17), and (3.18),

$$\begin{aligned} S_3 &\leq I_{\{\tilde{M}(\ell\hat{e}_1)<x\}} \sum_{k\,:\,|L''_k|>h_0} |L''_k|^{2\alpha} \\ &\leq I_{\{\tilde{M}(\ell\hat{e}_1)<x\}} \sum_{k\,:\,|L''_k|>h_0} y_0^{-1}|\mathcal{C}_{\nu(L''_k)}|^{2\alpha} \\ &\leq I_{\{\tilde{M}(\ell\hat{e}_1)<x\}} \sum_{\nu\in\xi''} y_0^{-1}|\mathcal{C}_\nu|^{2\alpha}, \end{aligned}$$

and hence, using that $|\xi''|$, the number of sites (boxes) in ξ'', cannot exceed $\tilde{M}(\ell\hat{e}_1)$, we have for any $\gamma < 1$,

$$\begin{aligned} \{S_3 > x\} &\subset \Big\{|\xi''| < x^\gamma \;\text{ and } \sum_{\nu\in\xi''} |\mathcal{C}_\nu|^{2\alpha} > y_0 x\Big\} \\ &\quad \cup \Big\{x^\gamma \leq |\xi''| \leq x \text{ and } \sum_{\nu\in\xi''} |\mathcal{C}_\nu|^{2\alpha} > y_0 x\Big\} \\ &\subset \Big\{\exists \nu \in [-x^\gamma, x^\gamma]^d \cap \mathbb{Z}^d \text{ with } |\mathcal{C}_\nu| > y_0^{1/(2\alpha)} x^{(1-\gamma)/(2\alpha)}\Big\} \\ &\quad \cup \Big\{\exists \xi \in \Xi_0 \text{ with } |\xi| \geq x^\gamma \text{ and } \frac{1}{|\xi|}\sum_{\nu\in\xi} |\mathcal{C}_\nu|^{2\alpha} > y_0\Big\}. \end{aligned} \tag{3.23}$$

But, for some constant $b > 0$, $P[|\mathcal{C}_\nu| > x] \leq \exp(-bx)$ for all x (see, e.g., [Gr]), so

$$\begin{aligned} &P\Big[\exists \nu \in [-x^\gamma, x^\gamma]^d \cap \mathbb{Z}^d \text{ with } |\mathcal{C}_\nu| > y_0^{1/(2\alpha)} x^{(1-\gamma)/(2\alpha)}\Big] \\ &\qquad\qquad \leq (2x^\gamma + 1)^d \exp(-b y_0^{1/(2\alpha)} x^{(1-\gamma)/(2\alpha)}). \end{aligned} \tag{3.24}$$

By Theorem 5 of [HoN2], provided y_0 is sufficiently large (depending only on d and the distribution of the $|\mathcal{C}_\nu|$, which in turn depends only on d), we also have for some $a > 0$

and a possibly smaller b:

$$P\Big[\exists \xi \in \Xi_0 \text{ with } |\xi| \geq x^\gamma \text{ and } \frac{1}{|\xi|}\sum_{\nu\in\xi} |\mathcal{C}_\nu|^{2\alpha} > y_0\Big] \;\leq\; a\exp(-bx^{\gamma/(2\alpha+2)}). \tag{3.25}$$

The exponents $(1-\gamma)/(2\alpha)$ in (3.24) and $\gamma/(2\alpha+2)$ in (3.25) are both made equal to κ_5 by taking $\gamma = (\alpha+1)/(2\alpha+1)$. For this choice of γ, combining (3.23), (3.24) and (3.25) gives that

$$P[S_3 > x] \;\leq\; C_1 \exp(-C_0 x^{\kappa_5}) \text{ for all } x \tag{3.26}$$

for possibly some larger C_1 and smaller C_0. Noting that $\kappa_5 < \kappa_1$, combining (3.19), (3.22), and (3.26) yields that

$$P[S''_\ell > 3x] \;\leq\; C_1 \exp(-C_0 x^{\kappa_5}) \text{ for all } x \geq C_1\ell,$$

which proves (3.10) for possibly larger C_1 and smaller C_0. Step 2 is completed as follows:

$$\begin{aligned} ES''_\ell \;&=\; \int_0^\infty P[S''_\ell > x]\,dx \\ &\leq\; C_1\ell + \int_{C_1\ell}^\infty C_1 \exp(-C_0 x^{\kappa_5})\,dx \\ &=\; C_1\ell + o(\ell) \text{ as } \ell \to \infty. \end{aligned}$$

Step 3. $\operatorname{Var} T_\ell \leq C_1\ell$ *for* $\ell > 1$. Steps 1 and 2 show that, for appropriate C_1, $\operatorname{Var} T''_\ell < C_1\ell$ for $\ell > 1$. Now

$$\begin{aligned} \operatorname{Std} T_\ell \;&\leq\; \operatorname{Std} T''_\ell + \operatorname{Std}|T'_\ell - T''_\ell| + \operatorname{Std}|T'_\ell - T_\ell| \\ &\leq\; {C_1}^{1/2}\ell^{1/2} + \operatorname{Std}|T'_\ell - T''_\ell| + \operatorname{Std}|T'_\ell - T_\ell|. \end{aligned}$$

It follows from (3.3) that $\operatorname{Std}|T'_\ell - T_\ell|$ is bounded in ℓ. On the other hand, since $0 \leq T'_\ell, T''_\ell \leq \ell^\alpha$, we have $|T'_\ell - T''_\ell| \leq \ell^\alpha I_{\{T'_\ell \neq T''_\ell\}}$ so, assuming (3.4) holds,

$$\operatorname{Var}|T'_\ell - T''_\ell| \;\leq\; E(|T'_\ell - T''_\ell|^2) \;\leq\; \ell^{2\alpha} P[T'_\ell \neq T''_\ell] \;=\; o(\ell) \text{ as } \ell \to \infty,$$

yielding that, for possibly larger C_1, $\operatorname{Var} T_\ell < C_1\ell$ for $\ell > 1$. In view of Lemma 3.2, (2.4) will be proved once we complete the:

Proof of (3.4). For an $a > 1$ (to be chosen momentarily), let $B(a\ell) = [-a\ell, a\ell]^d$. If B is any cube containing 0 and $\ell\hat{e}_1$ such that $\overline{r'} \subset B$, $\overline{r''} \subset B$, $Q \cap B = Q_\ell \cap B$, and no link on $\overline{r''}$ exceeds h_ℓ in length, then $T'_\ell = T''_\ell$. Hence

$$\begin{aligned} P[T'_\ell \neq T''_\ell] \;&\le\; P[\overline{r'} \not\subset B(a\ell)] + P[\overline{r''} \not\subset B(a\ell)] \\ &\quad + P[\exists \text{ an } (\epsilon/3^{\lfloor \ell \rfloor})\text{-box touching } B(a\ell) \text{ with two or more Poisson particles}] \\ &\quad + P[\exists \text{ a } \lambda\text{-box } \nu \text{ touching } B(a\ell) \text{ with } |\mathcal{C}_\nu| \geq {y_0}^{1/(2\alpha)} h_\ell]\,, \end{aligned} \tag{3.27}$$

where we used (3.16) and (3.18). First, we bound the term $P[\overline{r''} \not\subset B(a\ell)]$. If $\overline{r''} \not\subset B(a\ell)$, then either $\beta \not\subset B(a\ell/2)$ or else $\beta \subset B(a\ell/2)$ and for some $(r''_{i_1}, r''_{i_1+1}, \ldots, r''_{i_2}, \ldots, r''_{i_2})$ we have that $\overline{r''_{i_1}\, r''_{i_1+1}}$ exits an ϵ-box β_k on β, $r''_{i_2} \notin B(a\ell)$, and $\overline{r''_{i_3-1}\, r''_{i_3}}$ re-enters β_k. By the No Doubling Back Proposition (Lemma 5.5), in the latter case we must have that r''_{i_1+1} and r''_{i_3-1} are within Euclidean distance $16\epsilon\sqrt{d}$ of $\beta_k \subset B(a\ell/2)$ and also that $|r''_{i_1+1} - r''_{i_3-1}| \leq 33\epsilon\sqrt{d}$. It follows also (since r'' is minimizing) that $|r''_i - r''_{i+1}| \leq 33\epsilon\sqrt{d}$ for $i_1 < i < i_3 - 1$. These together would imply that there is a cluster of overlapping balls of radius $17\epsilon\sqrt{d}$ centered at particle locations in Q touching both $B(a\ell/2)$ and $B(a\ell)^c$. Since $17\epsilon\sqrt{d}$ is less than the critical continuum percolation radius R^*_c, this latter event occurs with probability bounded by $C_1 \exp(-C_0\ell)$ — a consequence of Theorem 3.5 and Lemma 3.3 of [MR] (here C_0 and C_1 depend on d, ϵ and a). It follows that

$$P[\overline{r''} \not\subset B(a\ell)] \;\le\; P[\beta \not\subset B(a\ell/2)] \;+\; C_1 \exp(-C_0\ell).$$

We take C_1 as in the rightmost expression of (3.22) and then for sufficiently large a, we have from the definitions of β and $\tilde{M}(\ell\hat{e}_1)$ that $\{\beta \not\subset B(a\ell/2)\} \subset \{\tilde{M}(\ell\hat{e}_1) > C_1\ell\}$ so, as in (3.22), $P[\beta \not\subset B(a\ell/2)] \leq C_1 \exp(-C_0\ell^{\kappa_1})$. Since $\kappa_1 \leq 1$, this yields $P[\overline{r''} \not\subset B(a\ell)] \leq C_1 \exp(-C_0\ell^{\kappa_1})$. The first term on the right side of (3.27) may be similarly bounded for a possibly larger a.

With a now fixed, there are $O(\ell^d 3^{\ell d})$ $(\epsilon/3^{\lfloor \ell \rfloor})$-boxes and $O(\ell^d)$ λ-boxes touching $B(a\ell)$. Since the probability that any particular $(\epsilon/3^{\lfloor \ell \rfloor})$-box has two or more Poisson particles in it is bounded by $(\epsilon/3^{\lfloor \ell \rfloor})^{2d}$, the third term on the right side of (3.27) is of order $\ell^d 3^{-\ell d} \leq C_1 \exp(-\ell)$ for possibly larger C_1. Finally, by our earlier choice of λ, the probability that any particular λ-box ν has $|\mathcal{C}_\nu| \geq y_0^{1/(2\alpha)} h_\ell$ is bounded by $\exp(-b y_0^{1/(2\alpha)} h_\ell)$ yielding that the fourth term in (3.27) is bounded by $C_1 \exp(-C_0 \ell^{1/(2\alpha)})$ for possibly larger C_1 and smaller C_0 since $h_1 > 0$. Collectively this proves (3.4) since $\kappa_3 = 1/(2\alpha) < \kappa_1 \leq 1$. ■

This completes the proof of (2.4). We finish the proof of Theorem 2.1 with:

Step 4. Proof of (2.5). Our strategy here is to invoke Lemma 5.6 for large ℓ, using $\mathcal{F}_m$, Δ_m, and U_m from the previous section, i.e.,

$$\Delta_m \;=\; E[T''_\ell|\mathcal{F}_m] - E[T''_\ell|\mathcal{F}_{m-1}] \text{ and } U_m \;=\; (T_\ell^{(m)} - T''_\ell)^2.$$

We also therefore take $S = S_\ell$ as given in (3.9). We presently show that the hypotheses of the lemma are satisfied for appropriate x_0, c, and γ.

First, we observe that $0 \leq T_\ell^{(m)} - T''_\ell \leq 2^\alpha h_\ell^\alpha$. The first inequality is trivial and the second follows from Lemma 5.3. Since $T_\ell^{(m)}$ is independent of $\mathcal{F}(B_m)$ we see that $E[T_\ell^{(m)}|\mathcal{F}_m] = E[T_\ell^{(m)}|\mathcal{F}_{m-1}]$. It follows that $|\Delta_m| \leq 2^\alpha h_\ell^\alpha$. We therefore take $c = 2^\alpha h_\ell^\alpha$ in Lemma 5.6.

Next, we verify that $E[\Delta_m^2|\mathcal{F}_{m-1}] \leq E[U_m|\mathcal{F}_{m-1}]$ as follows:

$$\begin{aligned}
E[\Delta_m^2|\mathcal{F}_{m-1}] \;&=\; E[(E[T''_\ell|\mathcal{F}_m] - E[T''_\ell|\mathcal{F}_{m-1}])^2|\mathcal{F}_{m-1}] \\
&\leq\; E[(E[T''_\ell|\mathcal{F}_m] - E[T_\ell^{(m)}|\mathcal{F}_m])^2|\mathcal{F}_{m-1}] \\
&=\; E[(E[T''_\ell - T_\ell^{(m)}|\mathcal{F}_m])^2|\mathcal{F}_{m-1}] \\
&\leq\; E[E[(T''_\ell - T_\ell^{(m)})^2|\mathcal{F}_m]|\mathcal{F}_{m-1}] \\
&=\; E[(T''_\ell - T_\ell^{(m)})^2|\mathcal{F}_{m-1}] \;=\; E[U_m|\mathcal{F}_{m-1}].
\end{aligned}$$

The first inequality uses (3.7) with $\mathcal{G} = \mathcal{F}_{m-1}$, $X = E[T''_\ell|\mathcal{F}_m]$, and $Y = E[T_\ell^{(m)}|\mathcal{F}_{m-1}] = E[T_\ell^{(m)}|\mathcal{F}_m]$. The second inequality follows from the conditional Jensen's inequality.

Additionally, by (3.9) and (3.10) and with $\kappa_2 = 1/(4\alpha+3) < \kappa_5 = 1/(4\alpha+2)$, we get

$$\begin{aligned} P[S > x] &\leq P[S''_\ell > 2^{-(2\alpha+1)}x] \\ &\leq C_1 \exp\big(-C_0(2^{-(2\alpha+1)}x)^{\kappa_5}\big) \text{ for } x \geq 2^{2\alpha+1}C_1\ell \qquad (3.28) \\ &\leq C_1 \exp(-x^{\kappa_2}) \text{ for } x \geq 2^{2\alpha+1}C_1\ell, \end{aligned}$$

where the last inequality holds for a possibly larger C_1. This gives (5.15) with $\gamma = \kappa_2$ and $x_0 = 2^{2\alpha+1}C_1\ell$.

Finally, we must have $x_0 \geq c^2 \geq 1$. The first inequality holds if $h_\ell \leq (2C_1\ell)^{1/(2\alpha)}$. Recalling that $h_\ell = \max(h_0, h_1\ell^{1/(2\alpha)})$ where h_0 has already been specified, we take $h_1 = (2C_1)^{1/(2\alpha)}$. We will then have $x_0 \geq c^2$ for ℓ large enough that $h_\ell = h_1\ell^{1/(2\alpha)}$. The second inequality ($c \geq 1$) is equivalent to $h_\ell \geq 1/2$ which holds since $h_0 \geq 1$.

Lemma 5.6 implies that there are constants C_0 and C_1 such that, for ℓ large enough that $h_\ell = h_1\ell^{1/(2\alpha)}$,

$$P[\,|T''_\ell - ET''_\ell| > x\sqrt{\ell}] \leq C_1 \exp(-C_0 x) \text{ for } x \leq C_0\ell^{\kappa_2},$$

which can be made to hold for all ℓ by increasing C_1. Now

$$|T''_\ell - T_\ell| \leq |T''_\ell - T'_\ell| + |T'_\ell - T_\ell| \leq \ell^\alpha I_{\{T''_\ell \neq T'_\ell\}} + |T'_\ell - T_\ell|,$$

so it follows from Lemma 3.1 that $|ET''_\ell - ET_\ell|$ is bounded by some constant $\tilde{b}$. Also, using that

$$|T_\ell - ET_\ell| \leq |T_\ell - T'_\ell| + |T'_\ell - T''_\ell| + |T''_\ell - ET''_\ell| + |ET''_\ell - ET_\ell|,$$

we get, for $\ell > 1$ and $\tilde{b} \leq x \leq C_0\ell^{\kappa_2}$, that

$$\begin{aligned} P[|T_\ell - ET_\ell| > 3x\sqrt{\ell}] &\leq P[|T_\ell - T'_\ell| > x\sqrt{\ell}] + P[T'_\ell \neq T''_\ell] \\ &\quad + P[|T''_\ell - ET''_\ell| > x\sqrt{\ell}] \\ &\leq C_1 \exp(-C_0(x\sqrt{\ell})^{\kappa_4}) + C_1 \exp(-C_0\ell^{\kappa_3}) \\ &\quad + C_1 \exp(-C_0 x). \qquad (3.29) \end{aligned}$$

On the one hand, (3.29) produces for appropriate C_0 and C_1 and for $\ell > 1$ and $\tilde{b} \le x \le C_0 \ell^{\kappa_2}$,

$$P[|T_\ell - ET_\ell| > 3x\sqrt{\ell}] \le C_1 \exp(-C_0 x^{\kappa_4}) + C_1 \exp(-C_0 x^{\kappa_3/\kappa_2}) + C_1 \exp(-C_0 x)$$
$$\le C_1 \exp(-C_0 x^{\kappa_1}),$$

with the last inequality holding for possibly larger C_1 since $\kappa_1 = \min(1, \kappa_4)$ and $\kappa_3/\kappa_2 > 1$. By possibly increasing C_1 still further and decreasing C_0 we can ensure that

$$P[|T_\ell - ET_\ell| > x\sqrt{\ell}] \ \le \ C_1 \exp(-C_0 x^{\kappa_1}) \text{ for all } \ell \text{ and } x \le C_0 \ell^{\kappa_2},$$

proving (2.5). ■

4. Proof of Theorems 2.2 and 2.3

Our plan is to show that ET_ℓ exhibits the following sort of weak superadditivity:

Lemma 4.1. *For some constant $C_1 \in (0, \infty)$ we have*

$$ET_{2\ell} \ \ge \ 2ET_\ell - C_1 \sqrt{\ell} (\log \ell)^{1/\kappa_1} \quad \textit{for all large } \ell. \tag{4.1}$$

Before proving this lemma, which constitutes the bulk of this section, we show how this gives Theorems 2.2 and 2.3. First, we need the following easy lemma; it will be applied with $a(\ell) = ET_\ell$ and $g(l) = C_1 \sqrt{\ell} (\log \ell)^{1/\kappa_1}$.

Lemma 4.2. *Suppose the functions $a : \mathbb{R}^+ \to \mathbb{R}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the following conditions: $a(\ell)/\ell \to \nu \in \mathbb{R}$, $g(\ell)/\ell \to 0$ as $\ell \to \infty$, $a(2\ell) \ge 2a(\ell) - g(\ell)$, and $\psi \equiv \limsup_{\ell \to \infty} g(2\ell)/g(\ell) < 2$. Then, for any $c > 1/(2 - \psi)$, $a(\ell) \le \nu \ell + c g(\ell)$ for all large ℓ.*

Proof. It is easily verified that, for $c > 1/(2-\psi)$, $\tilde{a}(\ell) \equiv a(\ell) - cg(\ell)$ satisfies $\tilde{a}(2\ell) \ge 2\tilde{a}(\ell)$ for all large ℓ. Iterating this n times yields $\tilde{a}(2^n \ell) \ge 2^n \tilde{a}(\ell)$ or $\tilde{a}(2^n \ell)/(2^n \ell) \ge \tilde{a}(\ell)/\ell$. Under our hypotheses on a and g, $\tilde{a}(x)/x \to \nu$ as $x \to \infty$, so letting $n \to \infty$ shows that $\tilde{a}(\ell)/\ell \le \nu$ for all large ℓ. ■

Proof of Theorems 2.2 and 2.3. Based on general subadditivity considerations, we have (see [HoN1]) that

$$0 \;<\; \mu \;\equiv\; \inf_{\ell>0} \frac{ET_\ell}{\ell} \;<\; \infty \quad \text{and} \quad \lim_{\ell\to\infty} \frac{T_\ell}{\ell} \;=\; \mu \text{ (a.s. and in } L^1). \tag{4.2}$$

Taking $a(\ell) = ET_\ell$ and $g(\ell) = C_1\sqrt{\ell}(\log \ell)^{1/\kappa_1}$ in Lemma 4.2 (so that $\limsup_\ell g(2\ell)/g(\ell) = \sqrt{2} < 2$), we get that, for appropriate C_1,

$$\mu\ell \;\le\; ET_\ell \;\le\; \mu\ell + C_1\sqrt{\ell}(\log \ell)^{1/\kappa_1} \text{ for large } \ell. \tag{4.3}$$

The second part of Theorem 2.1 then immediately implies that

$$P[|T_\ell - \mu\ell| > 2x\sqrt{\ell}] \;\le\; C_1 \exp(-C_0 x^{\kappa_1}) \quad \text{for } C_1(\log \ell)^{1/\kappa_1} \le x \le C_0 \ell^{\kappa_2}.$$

Substituting $\lambda = 2x\sqrt{\ell}$ yields (2.7) for large ℓ, with this latter restriction lifted by adjusting C_0 and C_1, which proves Theorem 2.2. On the other hand, substituting $x = \frac{1}{2}(\log \ell)^{(1+\epsilon)/\kappa_1}$, where $\epsilon > 0$, yields

$$P[|T_\ell - \mu\ell| > \sqrt{\ell}(\log \ell)^{(1+\epsilon)/\kappa_1}] \;\le\; C_1 \ell^{-C_0(\log \ell)^\epsilon} \quad \text{for large } \ell.$$

This and the Borel-Cantelli Lemma together imply that, a.s., the event $\{\big|T(0,w) - \mu|w|\big| > \sqrt{|w|}(\log |w|)^{(1+\epsilon)/\kappa_1}\}$ occurs for only finitely many $w \in \mathbb{Z}^d$. Theorem 2.3 follows from this together with an application of Lemma 5.2 and the Borel-Cantelli Lemma. Further details are left to the reader. ∎

Proof of Lemma 4.1. Fix γ with $0 < \alpha\gamma < 1/2$. Define the event

$$\tilde{F}_\ell \;\equiv\; \{\text{there exists an } x \in \mathbb{R}^d \text{ with } |x - \ell\hat{e}_1| \le 3\ell \text{ and } |q(x) - x| \ge \ell^\gamma\}.$$

Next, take $x_1 = \ell\hat{e}_1$ and pick $x_2, \ldots, x_{n(\ell)}$ on $\partial\mathcal{B}(0,\ell)$, the Euclidean sphere of radius ℓ centered at the origin, so that every $x \in \partial\mathcal{B}(0,\ell)$ is within (Euclidean) distance ℓ^γ of one of the x_i. We may arrange that $n(\ell) \le C_1\ell^{(1-\gamma)(d-1)}$ as the following constructive

sketch shows. Take $x_1 = \ell\hat{e}_1$ and suppose $x_1, \dots, x_k$ have already been selected. Choose $x_{k+1} \in \partial\mathcal{B}(0,\ell) \setminus (\cup_{i=1}^k \mathcal{B}(x_i, \ell^\gamma))$ if this latter set is non-empty, and stop otherwise. The Eulcidean balls $\mathcal{B}(x_i, \ell^\gamma/2)$ cover disjoint patches of $\partial\mathcal{B}(0,\ell)$ with $(d-1)$-dimensional area of order $\ell^{\gamma(d-1)}$. Since $\partial\mathcal{B}(0,\ell)$ has total area of order ℓ^{d-1}, it follows that the process must stop after order $\ell^{(1-\gamma)(d-1)}$ steps.

Also, take $x_i' = 2\ell\hat{e}_1 - x_i$ so each x_i' is on $\partial\mathcal{B}(2\ell\hat{e}_1, \ell)$ and every $x \in \partial\mathcal{B}(2\ell\hat{e}_1,\ell)$ is within distance ℓ^γ of one of the x_i'. The x_i' are simply the x_i radially reflected about $x_1 = \ell\hat{e}_1$ and they bear the same spatial relation to each other as do the x_i.

We claim that for some constant C_1, for large ℓ we have

$$T_{2\ell} \;\geq\; \min_{1\leq i\leq n(\ell)} T(0,x_i) \;+\; \min_{1\leq j\leq n(\ell)} T(2\ell\hat{e}_1, x_j') \;-\; C_1\ell^{\gamma\alpha} \quad \text{on } \tilde{F}_\ell^c. \tag{4.4}$$

To see this, let $r = (q_k)$ denote the path from $q(0)$ to $q(2\ell\hat{e}_1)$ that realizes $T_{2\ell}$. Let $\tilde{q} = q_{k^*}$ denote the first q_k on r not in $\mathcal{B}(0,\ell)$ and put $q = q_{k^*-1}$. (Since r ends with $q(2\ell\hat{e}_1)$ and, on $\tilde{F}_\ell^c$, $|q(2\ell\hat{e}_1) - 2\ell\hat{e}_1| < \ell^\gamma < \ell$, such a k^* exists; furthermore, $k^* \neq 0$ since r begins with $q(0)$ and $|q(0)| < \ell$ on $\tilde{F}_\ell^c$.) Similarly, let q' denote the first q_k on r such that q' and all subsequent q_k's on r lie within $\mathcal{B}(2\ell\hat{e}_1, \ell)$. Then clearly

$$T_{2\ell} \;\geq\; T(0,q) + T(q', 2\ell\hat{e}_1).$$

Now let $x = \overline{q\,\tilde{q}} \cap \partial\mathcal{B}(0,\ell)$; it follows from (5.3) of Lemma 5.2 that, for some C_1, on $\tilde{F}_\ell^c$ we must have $|q - x| \leq C_1\ell^\gamma$ for all large ℓ. Picking x_{i^*} so that $|x_{i^*} - x| \leq \ell^\gamma$, we get that

$$|q(x_{i^*}) - q| \;\leq\; |q(x_{i^*}) - x_{i^*}| + |x_{i^*} - x| + |x - q| \;\leq\; (2 + C_1)\ell^\gamma.$$

It follows that $T(0, x_{i^*}) \leq T(0,q) + (2+C_1)^\alpha \ell^{\alpha\gamma}$ and hence

$$T(0,q) \;\geq\; \min_{1\leq i\leq n(\ell)} T(0,x_i) - (2+C_1)^\alpha\ell^{\alpha\gamma}.$$

Similarly,

$$T(2\ell\hat{e}_1, q') \;\geq\; \min_{1\leq j\leq n(\ell)} T(2\ell\hat{e}_1, x_j') - (2+C_1)^\alpha\ell^{\alpha\gamma},$$

yielding (4.4) for an appropriately larger C_1. Since $x_1 = x_1' = \ell\hat{e}_1$, it follows that

$$\min_{1\le i\le n(\ell)} T(0,x_i) \;+\; \min_{1\le j\le n(\ell)} T(2\ell\hat{e}_1, x_j') \;\le\; T_{2\ell} + C_1\ell^{\alpha\gamma} + T(0,\ell\hat{e}_1)I_{\tilde{F}_\ell} + T(2\ell\hat{e}_1,\ell\hat{e}_1)I_{\tilde{F}_\ell}.$$

Taking expectations and using the symmetry of our construction together with the Cauchy-Schwarz inequality yields

$$2E[\min_i T(0,x_i)] \;\le\; ET_{2\ell} + C_1\ell^{\alpha\gamma} + 2\sqrt{E[T_\ell^2]P[\tilde{F}_\ell]}. \tag{4.5}$$

Now $E[T_\ell^2] = (ET_\ell)^2 + \operatorname{Var} T_\ell$, where the second summand is of order ℓ by Theorem 2.1 and the first term is of order ℓ^2 by general subadditivity arguments (see (7) in [HoN1]).

It follows from (5.2) of Lemma 5.2 (for possibly different C_0 and C_1) that $P[\tilde{F}_\ell] \le C_1\exp(-C_0\ell^{\gamma d})$. Hence

$$\sqrt{E[T_\ell^2]P[\tilde{F}_\ell]} \;=\; o(1) \;=\; o(\ell^{\alpha\gamma}) \text{ as } \ell\to\infty$$

and

$$\begin{aligned} ET_{2\ell} \;&\ge\; 2E[\min_{1\le i\le n(\ell)} T(0,x_i)] - C_1\ell^{\alpha\gamma} \\ &=\; 2ET_\ell \;-\; 2E[\max_{1\le i\le n(\ell)} \Big(E[T(0,x_i)] - T(0,x_i)\Big)] \;-\; C_1\ell^{\alpha\gamma}. \end{aligned}$$

The equality above uses that $E[T(0,x_i)] = E[T(0,x_1)] = ET_\ell$. Since $\alpha\gamma < 1/2$, Lemma 4.1 will be proved if we establish that

$$E[\max_{1\le i\le n(\ell)} \Big(E[T(0,x_i)] - T(0,x_i)\Big)] \;\le\; C_1\sqrt{\ell}(\log\ell)^{1/\kappa_1}. \tag{4.6}$$

To conclude the proof of Lemma 4.1, take $Y_i^{(\ell)} = T(0,x_i)/\sqrt{\ell}$ in Lemma 4.3 below and note that hypotheses are satisfied with $a = \frac{1}{2}+\epsilon$, $\tilde{a} = (1-\gamma)(d-1)+\epsilon$, $b=\kappa_1$, $\tilde{b}=\kappa_2$, and C_0 and C_1 as in Theorem 2.1.

Lemma 4.3. *For $\ell \ge \ell_0 > 1$, let $Y_i^{(\ell)}$ for $1\le i\le n(\ell)$ be non-negative random variables on a common probability space such that, for some $a,\tilde{a},b,\tilde{b},C_0,C_1 \in (0,\infty)$,*

$$E[Y_i^{(\ell)}] \;\le\; \ell^a \quad \textit{and} \quad n(\ell) \;\le\; \ell^{\tilde{a}}, \tag{4.7}$$

and,

$$P(|Y_i^{(\ell)} - E[Y_i^{(\ell)}]| > x) \;\le\; C_1 \exp(-C_0 x^b) \quad \text{for } x \le C_0 \ell^{\tilde{b}} \tag{4.8}$$

Then, for some $C_2 = C_2(\ell_0, a, \tilde{a}, b, \tilde{b}, C_0, C_1)$,

$$E[\max_{1\le i\le n(\ell)} (E[Y_i^{(\ell)}] - Y_i^{(\ell)})] \;\le\; C_2(\log \ell)^{1/b} \quad \text{for all } \ell \ge \ell_0. \tag{4.9}$$

Proof. Let $M^{(\ell)}$ denote $\max_{1\le i\le n(\ell)}(E[Y_i^{(\ell)}] - Y_i^{(\ell)})$ and put $f(\ell) = \hat{C}(\log \ell)^{1/b}$ where we take $\hat{C}$ so that $C_0\hat{C}^b = a + \tilde{a}$. Note that $M^{(\ell)} \le \ell^a$ since the $Y_i^{(\ell)}$ are non-negative, so

$$M^{(\ell)} \;\le\; \begin{cases} f(\ell) & \text{if } Y_i^{(\ell)} - E[Y_i^{(\ell)}] \ge -f(\ell) \text{ for all } i \le n(\ell) \\ \ell^a & \text{otherwise.} \end{cases}$$

For large ℓ, $f(\ell) \le C_0\ell^{\tilde{b}}$ and we have

$$\begin{aligned} E[M^{(\ell)}] \;&\le\; f(\ell) \;+\; \ell^a \sum_{i=1}^{n(\ell)} P\Big(Y_i^{(\ell)} - E[Y_i^{(\ell)}] \le -f(\ell)\Big) \\ &\le\; f(\ell) \;+\; \ell^{a+\tilde{a}} C_1 \exp(-C_0 f(\ell)^b) \\ &=\; f(\ell) \;+\; C_1 \;\le\; C_2(\log \ell)^{1/b}, \end{aligned}$$

where the equality follows from our choice of $\hat{C}$ and the final inequality holds for an appropriate C_2. The second inequality above holds only for large ℓ, but since $EM^{(\ell)} \le \ell^a$ we can ensure that $EM^{(\ell)} \le C_2(\log \ell)^{1/b}$ for all $\ell \ge \ell_0$ by making C_2 larger if necessary. ∎

5. Technical Lemmas

Throughout this section, ϕ is any cost function of the form

$$\phi(t) \;=\; \begin{cases} t^\alpha & \text{if } t \le h \\ h^\alpha + \alpha h^{\alpha-1}(t-h) & \text{otherwise,} \end{cases}$$

with $\alpha > 1$ and $h > 0$. Recall our notation that, for any cost function ϕ of this form and $a, b \in \mathbb{R}^d$,

$$\mathcal{W}_\phi(a,b) \;=\; \{c \in \mathbb{R}^d : \phi(|a-c|) + \phi(|c-b|) \le \phi(|a-b|)\} \tag{5.1}$$

and that $\mathcal{W}(a,b) = \mathcal{W}_{\phi_\infty}(a,b)$, where $\phi_\infty(t) = t^\alpha$. We provide below in Lemma 5.1 some elementary geometric properties of these regions.

Lemma 5.1. *The region $\mathcal{W}_\phi(0, \ell\hat{e}_1)$ is closed and convex, contains $\frac{1}{2}\ell\hat{e}_1$ in its interior, and is invariant with respect to rotations about the first coordinate axis. Also, $\mathcal{W}_\phi(a,b)$ is the set $\mathcal{W}_\phi(0, |a-b|\hat{e}_1)$ rigidly moved so that 0 is moved to a and $|a-b|\hat{e}_1$ is moved to b. (By the rotational invariance of $\mathcal{W}_\phi(0, |a-b|\hat{e}_1)$ about the first coordinate axis, any such rigid motion will do.) In the case $\phi = \phi_\infty$, $\mathcal{W}(0, \ell\hat{e}_1) = \ell\mathcal{W}(0, \hat{e}_1)$ and $\ell' < \ell$ implies that $\mathcal{W}(0, \ell'\hat{e}_1) \subset \mathcal{W}(0, \ell\hat{e}_1)$.*

Proof. Much of this Lemma is self-evident. We prove only the convexity claim and the statements about the case $\phi = \phi_\infty$. The convexity of $\mathcal{W}_\phi(0, \ell\hat{e}_1)$ follows from the facts that ϕ is convex and increasing as follows. For $c, c' \in \mathcal{W}_\phi(0, \ell\hat{e}_1)$, and $\lambda \in [0,1]$:

$$\begin{aligned}
\phi(\ell) &\geq \lambda(\phi(|c|) + \phi(|c - \ell\hat{e}_1|)) \; + \; (1-\lambda)(\phi(|c'|) + \phi(|c' - \ell\hat{e}_1|)) \\
&\geq \phi(\lambda|c| + (1-\lambda)|c'|) \; + \; \phi(\lambda|c - \ell\hat{e}_1| + (1-\lambda)|c' - \ell\hat{e}_1|) \\
&\geq \phi(|\lambda c + (1-\lambda)c'|) \; + \; \phi(|\lambda(c - \ell\hat{e}_1) + (1-\lambda)(c' - \ell\hat{e}_1)|) \\
&= \phi(|\lambda c + (1-\lambda)c'|) \; + \; \phi(|\lambda c + (1-\lambda)c' - \ell\hat{e}_1)|),
\end{aligned}$$

so also $\lambda c + (1-\lambda)c' \in \mathcal{W}_\phi(0, \ell\hat{e}_1)$. That $\mathcal{W}(0, \ell\hat{e}_1) = \ell\mathcal{W}(0, \hat{e}_1)$ follows from the (degree α) homogeneity of ϕ_∞. If $\ell' < \ell$, $\mathcal{W}(0, \ell'\hat{e}_1) = \ell'\mathcal{W}(0, \hat{e}_1) \subset \ell\mathcal{W}(0, \hat{e}_1) = \mathcal{W}(0, \ell\hat{e}_1)$, where the containment follows since 0 is in the convex $\mathcal{W}(0, \hat{e}_1)$. ■

Lemma 5.2. *For $\gamma \in (0,1)$, let $A_{\gamma,\ell} \equiv \{\exists\, a \in \mathbb{R}^d \text{ with } |a| \leq 2\ell \text{ and } |a - q(a)| \geq \ell^\gamma\}$. Then, for some C_0 and C_1:*

$$P[A_{\gamma,\ell}] \leq C_1 \exp(-C_0 \ell^{\gamma d}), \tag{5.2}$$

and furthermore, for large ℓ, on $A^c_{\gamma,\ell}$,

$$\sup\{|a-b| : |a| \leq \ell, \; b \in \mathbb{R}^d, \; \mathcal{W}(a,b) \cap Q = \emptyset\} \; \leq \; C_1\ell^\gamma. \tag{5.3}$$

Remark. If $\Gamma \equiv \sup\{|a| : a \in \mathbb{R}^d, \; \mathcal{W}(0,a) \cap Q = \emptyset\}$, then (for large ℓ) $\Gamma \leq C_1\ell^\gamma$ on $A^c_{\gamma,\ell}$. By the substitution $x = C_1\ell^\gamma$, it follows that

$$P[\Gamma > x] \; \leq \; C_1 \exp(-C_0 x^d) \tag{5.4}$$

(for possibly different C_0 and C_1). Also, on $A^c_{\gamma,\ell}$, if (q,q') is any geodesic segment with $|q| \le \ell$ (or $|q'| \le \ell$), then $|q-q'| \le C_1\ell^\gamma$. It follows (for possibly different C_0 and C_1) that

$$P[\exists \text{ geodesic segment } (q,q') \text{ with } |q| \le \ell \text{ or } |q'| \le \ell \text{ and } |q-q'| > \ell^\gamma] \ \le\ C_1 \exp(-C_0\ell^{\gamma d}). \tag{5.5}$$

While Lemma 5.2 gives (5.4) and (5.5) for large x and ℓ, respectively, this restriction is removed by increasing C_1.

Proof of Lemma 5.2. For large ℓ, we have that

$$A_{\gamma,\ell} \ \subset\ \{\exists\ a \in \mathbb{Z}^d \text{ with } |a| \le 2\ell \text{ and } |a - q(a)| \ge \ell^\gamma/2\}.$$

This larger event has probability bounded by $C_1\ell^d \exp(-C_0\ell^{\gamma d})$, which, for smaller C_0 is bounded (for large ℓ) by $C_1 \exp(-C_0\ell^{\gamma d})$. By increasing C_1 if necessary, (5.2) will hold for all ℓ.

To get (5.3), we take C_1 large enough so that

$$\mathcal{B}(\frac{1}{2}\hat{e}_1, {C_1}^{-1}) \ \subset\ \mathcal{W}(0,\hat{e}_1). \tag{5.6}$$

Suppose ℓ is large enough so that $\ell > C_1\ell^\gamma$ and so that, for a configuration Q, we can find $a, b \in \mathbb{R}^d$ satisfying: $|a| \le \ell$, $|a-b| > C_1\ell^\gamma$, with $\mathcal{W}(a,b)$ devoid of particles from Q. If $|b| < 2\ell$, but $\tilde{b} = b$; otherwise put $\tilde{b} = \overline{a\,b} \cap \partial\mathcal{B}(0,2\ell)$. Then, since $|a-\tilde{b}| \ge \ell$ and $\mathcal{W}(a,\tilde{b}) \subset \mathcal{W}(a,b)$ (by Lemma 5.1), a and $\tilde{b}$ satisfy: $|(a+\tilde{b})/2| < 2\ell$, $|a-\tilde{b}| > C_1\ell^\gamma$, with $\mathcal{W}(a,\tilde{b})$ devoid of Poisson particles. Since, using (5.6), $\mathcal{B}((a+\tilde{b})/2, {C_1}^{-1}|a-\tilde{b}|) \subset \mathcal{W}(a,\tilde{b})$, it follows that

$$\left| q\Big(\frac{a+\tilde{b}}{2}\Big) - \frac{a+\tilde{b}}{2} \right| \ \ge\ {C_1}^{-1}|a-\tilde{b}| \ >\ \ell^\gamma,$$

i.e., the configuration Q belongs to $A_{\gamma,\ell}$. ■

Lemma 5.3. *For any $a, b, c \in \mathbb{R}^d$ we have:*

$$\phi^2(|a-c|) \ \le\ 2^{2\alpha}(\phi^2(|a-b|) + \phi^2(|b-c|)) \tag{5.7}$$

and

$$\phi(|a-c|) - \phi(|a-b|) - \phi(|b-c|) \ \le \ 2^\alpha h^\alpha. \tag{5.8}$$

Proof. First we prove (5.7). If $t \le 2h$ then

$$\frac{\phi(t)}{\phi(t/2)} \ = \ \frac{\phi(t)}{(t/2)^\alpha} \ \le \ \frac{t^\alpha}{(t/2)^\alpha} \ = \ 2^\alpha.$$

If $t > 2h$, put $t = (1+y)2h$ where $y > 0$. Then

$$\frac{\phi(t)}{\phi(t/2)} \ = \ \frac{1+\alpha(1+2y)}{1+\alpha y} \ = \ 1+\alpha\frac{1+y}{1+\alpha y} \ \le \ 1+\alpha \ \le \ 2^\alpha,$$

with the latter two inequalities holding since $\alpha > 1$. Thus $\phi(t) \le 2^\alpha \phi(t/2)$ for all $t \ge 0$. Now suppose, without loss of generality, that $|a-b| \le |b-c|$ so $|b-c| \ge \frac{1}{2}|a-c|$ and

$$\phi(|a-c|) \ \le \ 2^\alpha \phi(\frac{1}{2}|a-c|) \ \le \ 2^\alpha \phi(|b-c|)$$

giving that

$$\phi^2(|a-c|) \ \le \ 2^{2\alpha}\phi^2(|b-c|) \ \le \ 2^{2\alpha}(\phi^2(|a-b|) + \phi^2(|b-c|)),$$

and verifying (5.7). To establish (5.8), we first show by examining cases that, for $x, y \ge 0$,

$$\phi(x+y) - \phi(x) - \phi(y) \le 2^\alpha h^\alpha. \tag{5.9}$$

This clearly holds if $x+y \le 2h$. If $x+y > 2h$ with $x > h$ and $y \le h$, then

$$\phi(x+y) - \phi(x) - \phi(y) \ \le \ \phi(x+y) - \phi(x) \ = \ \alpha h^{\alpha-1} y \ \le \ \alpha h^\alpha \ \le \ 2^\alpha h^\alpha.$$

A symmetric argument works for $x+y > 2h$ with $x \le h$ and $y > h$. Finally, if $x > h$ and $y > h$, then

$$\phi(x+y) - \phi(x) - \phi(y) \ = \ (\alpha-1)h^\alpha \ \le \ 2^\alpha h^\alpha.$$

To complete the proof of (5.8), let b' be the orthogonal projection of b onto the line passing through a and c. Then the left side of (5.8) is dominated by $\phi(|a-c|) - \phi(|a-b'|) - \phi(|b'-c|)$. If $b' \notin \overline{ac}$, this quantity is negative. If $b' \in \overline{ac}$, then (5.9) yields (5.8). ∎

Lemma 5.4. *For any $E > 0$ and $a, b \in \mathbb{R}^d$, let $\mathcal{H}_E(a,b)$ denote the set*

$$\begin{aligned}\mathcal{H}_E(a,b) \;=\; &\{c \in \mathbb{R}^d : \exists \text{ a point } p \text{ on the line segment connecting} \\ &\qquad \tfrac{3}{4}a + \tfrac{1}{4}b \text{ and } \tfrac{1}{4}a + \tfrac{3}{4}b \text{ such that } |c-p| \le E\},\end{aligned}$$

and define $\mathcal{W}_\phi(a,b)$ as in (5.1). Then for any $E > 0$, there is an $h_0 > 0$ such that $\mathcal{H}_E(a,b) \subset \mathcal{W}_\phi(a,b)$ whenever $|a-b| > h_0$ and $h > h_0$.

Proof. Clearly it suffices to prove this for $a = 0$ and $b = \ell\hat{e}_1$ where $\ell > 0$. Let c be any point whose $\hat{e}_1$ coordinate is $\ell/2$ and put $u = |c - (\ell/2)\hat{e}_1|$. First, by examining cases, we calculate how large u may be while keeping c inside $\mathcal{W}_\phi(0, \ell\hat{e}_1)$. Since $|c| = |c - \ell\hat{e}_1|$, to have $c \in \mathcal{W}_\phi(0,\ell\hat{e}_1)$ we need $2\phi(|c|) \le \phi(\ell)$ for which it is sufficient to have:

$$2\phi(\frac{\ell}{2} + u) \;\le\; \phi(\ell). \tag{5.10}$$

If $\ell < h$, to have (5.10), it suffices to have $2(\frac{\ell}{2}+u)^\alpha \le \ell^\alpha$ or

$$u \le (2^{-1/\alpha} - 2^{-1})\ell. \tag{5.11}$$

On the other hand, if $\ell > 2h$, (5.10) will obtain provided

$$2(h^\alpha + \alpha h^{\alpha-1}(\frac{\ell}{2} + u - h)) \;\le\; h^\alpha + \alpha h^{\alpha-1}(\ell - h),$$

which reduces to:

$$u \;\le\; \frac{\alpha-1}{2\alpha}h. \tag{5.12}$$

Finally, if $h \le \ell \le 2h$, it suffices to have

$$2\Big(\frac{\ell}{2} + u\Big)^\alpha \;\le\; h^\alpha + \alpha h^{\alpha-1}(\ell - h),$$

or, equivalently,

$$u \;\le\; \left[2^{-1/\alpha}(1 + \alpha(\frac{\ell}{h} - 1))^{1/\alpha} - \frac{1}{2}\frac{\ell}{h}\right]h.$$

One verifies by calculus that the quantity in brackets, viewed as a function of ℓ, is increasing on the interval $[h, \frac{\alpha+1}{\alpha}h]$ and decreasing on $[\frac{\alpha+1}{\alpha}h, 2h]$. It therefore suffices for the case $h \leq \ell \leq 2h$ to have

$$u \ \leq \ \min(2^{-1/\alpha} - 2^{-1}, 2^{-1/\alpha}(1+\alpha)^{1/\alpha} - 1)h. \tag{5.13}$$

(Note that this minimum is strictly greater than 0 since $\alpha > 1$.) Using (5.11), (5.12), and (5.13), we see that to ensure that $c \in \mathcal{W}_\phi(0, \ell\hat{e}_1)$ it suffices to have

$$\begin{aligned} u \ &\leq \ C\min(\ell, h), \text{ where} \\ C \ &= \ \min(\frac{\alpha-1}{2\alpha},\ 2^{-1/\alpha} - 2^{-1},\ 2^{-1/\alpha}(1+\alpha)^{1/\alpha} - 1). \end{aligned}$$

That is,

$$\mathcal{U}_E(\ell) \ = \ \{c \in \mathbb{R}^d : c\text{'s } \hat{e}_1 \text{ coordinate is } \ell/2, \text{ and } |c - (\ell/2)\hat{e}_1| \leq E\}$$

satisfies $\mathcal{U}_E(\ell) \subset \mathcal{W}_\phi(0, \ell\hat{e}_1)$ if $E \leq C\min(\ell, h)$. It follows from the convexity of $\mathcal{W}_\phi(0, \ell\hat{e}_1)$ that the suspension of $\mathcal{U}_E(\ell)$ defined by

$$\mathcal{S}_E(\ell) \ = \ \{\rho\mathcal{U}_E(\ell) : 0 \leq \rho \leq 1\} \cup \{\rho\mathcal{U}_E(\ell) + (1-\rho)\ell\hat{e}_1 : 0 \leq \rho \leq 1\},$$

also satisfies

$$\mathcal{S}_E(\ell) \ \subset \mathcal{W}_\phi(0, \ell\hat{e}_1) \text{ for } E \leq C\min(\ell, h). \tag{5.14}$$

Elementary geometric arguments show that $\mathcal{H}_E(\ell) \subset \mathcal{S}_{4E}(\ell)$ if $\ell \geq 8E$. It follows from this and (5.14) that

$$\mathcal{H}_E(\ell) \ \subset \ \mathcal{W}_\phi(0, \ell\hat{e}_1) \text{ for } \ell \geq 8E \text{ and } (C/4)\min(\ell, h) \geq E,$$

proving the lemma for $h_0 = \max(8E, 4E/C)$. ■

The next purely geometric lemma (proved in [Ho]) states, roughly speaking that if $(q_0, \ldots, q_n)$ is a minimizing path with respect to the cost function ϕ and a segment $L = \overline{q_i\, q_{i+1}}$ passes near a segment $L' = \overline{q_{i'}\, q_{i'+1}}$ where $i < i'$, then this must happen near the end of L and the beginning of L'. Specifically:

Lemma 5.5 (No Doubling Back Proposition [Ho]). *Under the above arrangement, if* $a \in L$ *and* $b \in L'$*, then* $|q_{i+1} - a| \leq 16|a - b|$ *and* $|q_{i'} - b| \leq 16|a - b|$*. Also, therefore,* $|q_{i+1} - q_{i'}| \leq 33|a - b|$.

The following lemma is a modification of Theorem 3 of [Ke2].

Lemma 5.6. *Let* $(M_k : k \geq 0)$*,* $M_0 \equiv 0$*, be a martingale with respect to the filtration* $\mathcal{F}_k \uparrow \mathcal{F}$*. Put* $\Delta_k = M_k - M_{k-1}$ *and suppose* $(U_k : k \geq 1)$ *is a sequence of* $\mathcal{F}$*-measurable positive random variables satisfying* $E[\Delta_k^2 | \mathcal{F}_{k-1}] \leq E[U_k | \mathcal{F}_{k-1}]$*. With* $S = \sum_{k=1}^{\infty} U_k$*, suppose further that for finite constants* $C_1' > 0$*,* $0 < \gamma \leq 1$*,* $c \geq 1$*, and* $x_0 \geq c^2$ *we have* $|\Delta_k| \leq c$ *and*

$$P[S > x] \; \leq \; C_1' \exp(-x^\gamma), \;\; \text{when } x \geq x_0. \tag{5.15}$$

Then $\lim_{k \to \infty} M_k = M$ *exists and is finite almost surely and there are constants (not depending on* c *and* x_0*)* $C_2 = C_2(C_1', \gamma) < \infty$ *and* $C_3 = C_3(\gamma) > 0$ *such that*

$$P[|M| \geq x\sqrt{x_0}\,] \; \leq \; C_2 \exp(-C_3 x) \;\; \text{when } x \leq x_0^\gamma.$$

Proof. The proof of this lemma largely parallels the proof of Theorem 3 of [Ke2].

Throughout the proof, $C_2(C_1', \gamma)$ will denote a constant whose value depends only on C_1' and γ. As the proof progresses, C_2 will be made possibly larger several times, each occurrence of which is indicated by a "+" superscript: $C_2^+(C_1', \gamma)$. Similarly, $C_3(\gamma)$ will be made possibly smaller when indicated by a "−" superscript.

Following Kesten, put

$$\begin{aligned} A \; &= \; \sum_{k=1}^{\infty} E[\Delta_k^2 | \mathcal{F}_{k-1}], \\ \nu \; &= \; \inf\left\{\ell : \sum_{k=\ell+1}^{\infty} E[U_k | \mathcal{F}_\ell] > z\right\} \text{ (where } \inf\emptyset = \infty\text{), and} \\ \tilde{A} \; &= \; \sum_{k=1}^{\nu} E[\Delta_k^2 | \mathcal{F}_{k-1}]. \end{aligned}$$

Here $z > 0$ is arbitrary, but a specific choice will be made later. Then it follows exactly as in Kesten's Step 2 that

$$P[A \geq y] \;\leq\; P[\nu < \infty] \;+\; P[\tilde{A} \geq y] \tag{5.16}$$

and that, for any positive integer r, $E[\tilde{A}^r] \;\leq\; r! z^{r-1} ES$. Next, we estimate

$$ES \;=\; \int_0^\infty P[S > s]\, ds \;\leq\; x_0 + C_1' \int_0^\infty \exp(-s^\gamma)\, ds \;=\; x_0 + C_2(C_1', \gamma),$$

so $E[\tilde{A}^r] \;\leq\; r! z^{r-1}(x_0 + C_2(C_1', \gamma))$. Also, as in Kesten's (5.8), by taking $r = \lfloor y/z \rfloor$ where $y \geq z$, we get

$$\begin{aligned} P[\tilde{A} \geq y] \;&\leq\; C' \cdot (x_0 + C_2(C_1', \gamma)) \frac{1}{z} \exp(-\frac{y}{2z}) \\ &\leq\; C_2^+(C_1', \gamma) \exp(-\frac{y}{2z}) \end{aligned} \tag{5.17}$$

with the second inequality holding for $y \geq z \geq x_0$ since also $x_0 \geq 1$. (C' comes from Stirling's formula and the fact that $\sqrt{y/z} \leq \text{constant} \cdot \exp(y/2z)$.)

Next, as in Kesten's Step 3, we estimate $P[\nu < \infty]$. Let $S_m = \sum_{k=1}^m U_k$ and $S_{m,\ell} = E[S_m | \mathcal{F}_\ell]$. If $g(s) = \exp(\frac{1}{2}s^\gamma)$ then $g'(s) = \frac{1}{2}\gamma s^{\gamma-1} \exp(\frac{1}{2}s^\gamma) > 0$ for $s > 0$ and $g''(s) = \frac{1}{2}\gamma s^{\gamma-2} \exp(\frac{1}{2}s^\gamma)(\frac{1}{2}\gamma s^\gamma + \gamma - 1) > 0$ when $s^\gamma > 2(1-\gamma)/\gamma = 2\beta$. Hence $\tilde{g}(s) = (e^\beta \vee \exp(\frac{1}{2}s^\gamma))$ is convex giving that $\big(\tilde{g}(S_{m,\ell}) : \ell \geq 0\big)$ is a submartingale. Also, for $z \geq z(\gamma) = (2\beta)^{1/\gamma}$, $\tilde{g}(s) > \exp(\frac{1}{2}z^\gamma)$ if and only if $s > z$. So, for $z \geq z(\gamma)$:

$$\begin{aligned} P[\nu < \infty] \;&\leq\; \lim_{m\to\infty} \lim_{n\to\infty} P[\max_{\ell \leq n} S_{m,\ell} > z] \\ &=\; \lim_{m\to\infty} \lim_{n\to\infty} P\Big[\max_{\ell \leq n} \big\{\tilde{g}(S_{m,\ell})\big\} > \exp(\frac{1}{2}z^\gamma)\Big] \\ &\leq\; \limsup_{m\to\infty} \limsup_{n\to\infty} \exp(-\frac{1}{2}z^\gamma)\, E[\tilde{g}(S_{m,n})] \text{ (by Doob's inequality)} \\ &\leq\; \limsup_{m\to\infty} \limsup_{n\to\infty} \exp(-\frac{1}{2}z^\gamma)\, E[E\{\tilde{g}(S_m) | \mathcal{F}_n\}] \text{ (by Jensen's inequality)} \\ &\leq\; \limsup_{m\to\infty} \exp(-\frac{1}{2}z^\gamma)\, \big(e^\beta + E[g(S_m)]\big) \\ &=\; \exp(-\frac{1}{2}z^\gamma)\, \big(e^\beta + E[g(S)]\big). \end{aligned}$$

Now

$$
\begin{aligned}
E[g(S)] &\leq g(x_0)P[S \leq x_0] \;-\; \int_{x_0}^{\infty} g(s)\, dP[S > s] \\
&= g(x_0) \;+\; \int_{x_0}^{\infty} g'(s)P[S > s]\, ds, \\
&\leq \exp(\frac{1}{2}x_0^{\gamma}) \;+\; C_1' \frac{\gamma}{2} \int_{x_0}^{\infty} \exp(-\frac{1}{2}s^{\gamma})\, ds \quad (\text{since } s^{\gamma-1} \leq 1 \text{ on } [x_0, \infty)) \\
&\leq \exp(\frac{1}{2}x_0^{\gamma}) \;+\; C_2^{+}(C_1', \gamma) \;\; (\text{by replacing } \int_{x_0}^{\infty} \text{ with } \int_0^{\infty}).
\end{aligned}
$$

Hence, for $z > z(\gamma)$,

$$
\begin{aligned}
P[\nu < \infty] &\leq \exp(-\frac{1}{2}z^{\gamma})\big(C_2^{+}(C_1', \gamma) + \exp(\frac{1}{2}x_0^{\gamma})\big) \\
&\leq C_2^{+}(C_1', \gamma) \exp\big(-\frac{1}{2}(z^{\gamma} - x_0^{\gamma})\big).
\end{aligned} \tag{5.18}
$$

Following Kesten again by letting $y \to \infty$ and then $z \to \infty$, (5.17), (5.18), and (5.16) give that $P[A = \infty] = 0$. But $\lim_{k\to\infty} M_k = M$ exists and is finite almost surely on $\{A < \infty\}$ (See, e.g., Theorem 4.8 of [D].)

Next, as in Step 1 of Kesten and pp. 154-155 of [Ne] (this is where the boundedness of the martingale differences is used), for $y \geq cx > 0$:

$$
P[M \geq x] \;\leq\; P[A \geq y] \;+\; \exp(-\frac{x^2}{2ey}). \tag{5.19}
$$

Combining (5.16), (5.17), (5.18), and (5.19), we get that

$$
P[M \geq x] \;\leq\; C_2^{+}(C_1', \gamma)\Big[\exp(-\frac{z^{\gamma} - x_0^{\gamma}}{2}) + \exp(-\frac{y}{2z}) + \exp(-\frac{x^2}{2ey})\Big]
$$

whenever

$$
y \geq cx, \;\; y \geq z \geq x_0, \;\; \text{and } z \geq z(\gamma). \tag{5.20}
$$

Now, like in Kesten's Step 4, take $z = (x_0^{\gamma} + x^a)^{1/\gamma}$ where $a = 2\gamma/(1+2\gamma)$, and $y = xz^{1/2}$. Then $2z^{1/2} \leq 2^{1/\gamma}(x_0^{1/2} + x^{a/(2\gamma)})$ so, with $C_3(\gamma) = 2^{-1/\gamma}/e$,

$$
\frac{y}{2z} = \frac{x}{2z^{1/2}} \geq C_3(\gamma)\frac{x}{x_0^{1/2} + x^{a/(2\gamma)}} \text{ and } \frac{x^2}{2ey} = \frac{x}{2ez^{1/2}} \geq C_3(\gamma)\frac{x}{x_0^{1/2} + x^{a/(2\gamma)}}.
$$

Also, since $a = 1 - a/(2\gamma)$ and $C_3(\gamma) < 1/2$,

$$(z^\gamma - x_0^\gamma)/2 \;=\; x^a/2 \;=\; \frac{x/2}{x^{a/(2\gamma)}} \;\geq\; C_3(\gamma)\frac{x}{x_0^{1/2} + x^{a/(2\gamma)}}.$$

Presently we verify that for some constant $C_4(\gamma)$, (5.20) holds provided $x \geq C_4(\gamma)\sqrt{x_0}$. The relation $y \geq cx$ is equivalent to $z \geq c^2$; but $z \geq x_0 \geq c^2$, giving two inequalities in (5.20). To get $z \geq z(\gamma) = (2\beta)^{1/\gamma}$, it suffices to have $x \geq (2\beta)^{1/a}$ which, since $x_0 \geq c^2 \geq 1$, will hold if $x \geq (2\beta)^{1/a}\sqrt{x_0}$. Finally, $y \geq z$ is equivalent to $x^{2\gamma} \geq x_0^\gamma + x^a$ which will hold provided

$$\frac{1}{2}x^{2\gamma} \geq x_0^\gamma \text{ and } \frac{1}{2}x^{2\gamma} \geq x^a,$$

or, equivalently, when

$$x \geq 2^{1/2\gamma}\sqrt{x_0} \text{ and } x \geq 2^{1/(2\gamma-a)}. \tag{5.21}$$

Since $1/(2\gamma - a) = (1+2\gamma)/4\gamma^2 \geq 1/2\gamma$ and $x_0 \geq 1$, both conditions in (5.21) will hold provided $x \geq 2^{(1+2\gamma)/4\gamma^2}\sqrt{x_0}$. It therefore suffices to take $C_4(\gamma) = \max((2\beta)^{1/a}, 2^{(1+2\gamma)/(4\gamma^2)})$.

Letting $d = d(\gamma) = 2\gamma + 1$, we get

$$P[M \geq x] \;\leq\; C_2^+(C_1', \gamma)\exp\left[-C_3(\gamma)\frac{x}{x_0^{1/2} + x^{1/d}}\right]$$

whenever $x \geq C_4(\gamma)\sqrt{x_0}$. Now, for $C_4(\gamma)x_0^{1/2} \leq \tilde{x} \leq x_0^{d/2}$ we also have $\tilde{x}^{1/d} \leq x_0^{1/2}$, so

$$P[M \geq \tilde{x}] \;\leq\; C_2(C_1', \gamma)\exp\left[-C_3^-(\gamma)\frac{\tilde{x}}{\sqrt{x_0}}\right].$$

Substituting $\tilde{x} = x\sqrt{x_0}$, we get that, for $C_4(\gamma) \leq x \leq x_0^\gamma$,

$$P[M \geq x\sqrt{x_0}\,] \;\leq\; C_2(C_1', \gamma)\exp[-C_3(\gamma)x].$$

But for $x < C_4(\gamma)$, the exponential is bounded away from zero by $\exp(-C_3(\gamma)C_4(\gamma))$. Hence,

$$P[M \geq x\sqrt{x_0}\,] \;\leq\; C_2^+(C_1', \gamma)\exp[-C_3(\gamma)x] \text{ for } x \leq x_0^\gamma.$$

The lemma follows by a further application of this to the martingale $(-M_k : k \geq 0)$. ■

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